# Reconciling noninterference and gradual typing

Full Version with Definitions and Proofs

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## Abstract

One of the standard correctness criteria for gradual typing is the *dynamic gradual guarantee*, which ensures that loosening type annotations in a program does not affect its behavior in arbitrary ways. Though natural, prior work has pointed out that the guarantee does not hold of any gradual type system for information-flow control. Toro et al.'s $\mathrm{GSL}_{\mathsf{Ref}}$ language, for example, had to abandon it to validate noninterference.

We show that we can solve this conflict by avoiding a feature of prior proposals: *type-guided classification*, or the use of type ascription to classify data. Gradual languages require run-time secrecy labels to enforce security dynamically; if type ascription merely checks these labels without modifying them (that is, without classifying data), it cannot violate the dynamic gradual guarantee. We demonstrate this idea with *GLIO*, a gradual type system based on the LIO library that enforces both the gradual guarantee and noninterference, featuring higher-order functions, general references, coarse-grained information-flow control, security subtyping and first-class labels. We give the language a domain-theoretic semantics, using Pitts' framework of relational structures to prove noninterference and the dynamic gradual guarantee.

***CCS Concepts:*** **• Security and privacy → Information flow control**; **• Software and its engineering → Semantics**.

***Keywords:*** Gradual Typing, Noninterference

## 1 Introduction

Gradual type systems allow incomplete type annotations for combining the safety of static typing with the flexibility of dynamic languages. In the gradual $\lambda$-calculus of Siek and Taha [26], for example, we can declare the argument of a function $f$ as an integer but omit its return type. This causes the type checker to reject an expression such as $f(\mathtt{true})$ while accepting $f(0) + 1$, understanding that the latter will trigger a run-time error if $f(0)$ returns a string. Many language features have been adapted to gradual typing, including references [28], polymorphism [2, 16, 20, 33], among others.

Unlike other approaches that mix static and dynamic typing, ascribing types in a gradual language should barely affect a program's behavior, a property known as the *dynamic gradual guarantee* (DGG) [27]: the program might be rejected by the type checker or encounter more cast errors, but its output should not change from 0 to 1. Albeit natural, this isolation can be challenging for languages that strive for more than basic type safety. It had to be abandoned in a gradual variant of System F to enforce parametricity [33],[1] and in the $\mathrm{GSL}_{\mathsf{Ref}}$ language [32] to enforce noninterference. Sadly, the guarantee does not hold in any existing gradual language for information-flow control (IFC) [32].

```
let f x =
  let b (* : Bool<S> *) = true  in
  let y = ref b in
  let z = ref b in
  if  x then y := false else ();
  if !y then z := false else ();
  !z

f (<S>true)
```

**Figure 1.** Prototypical failure of the DGG due to NSU checks. The program throws an error when run, but successfully terminates if we uncomment the type annotation `Bool<S>`.

The goal of this paper is to remedy the situation for IFC languages without giving up on noninterference. The difficulty, we argue, stems from what we call *type-guided classification*: the ability to classify values through static type annotations. This issue is illustrated in Figure 1, which shows a program in $\lambda^{info}$ [4], a typical language for dynamic IFC. Values in $\lambda^{info}$ carry a *confidentiality label* that is checked and propagated during execution to prevent information leaks. Unannotated values such as `true` are marked with a default label (in $\lambda^{info}$, `Public`), which can be overridden with `< >`. For example, the function `f` is given a Secret argument.

For now, ignore the commented type `(* ... *)`. If we ran this program in a typical language with no IFC checks, it would have the effect of leaking the secret input `x` through the reference `z`, returning `true` when `x = true` and `false` when `x = false`. Dynamic IFC prevents this breach with a discipline known as *no-sensitive-upgrade* (NSU) [4, 5, 30], which forbids updates to public references when the control flow is influenced by secrets. In `f`, the reference `y` is implicitly labeled public because it is allocated in a public context and initialized with a public variable. This causes the NSU check to fail and terminate execution.

An extension of $\lambda^{info}$ with gradual types could allow us to annotate `b` with the type in comments, declaring it as a secret boolean. What would this declaration mean? Current

[1] Recent work has managed to lift this restriction using ideas similar to ours [20]; cf. Section 7.

gradual IFC languages ($GSL_{Ref}$ [32], ML-GS [11], etc.) interpret it as classification, thus setting b's dynamic secrecy label to S. This causes the program to terminate successfully: b's label is propagated to y and z, the program accepts the two assignments (because x has the same secrecy as the references), and returns <S>true. Unfortunately, this behavior violates the DGG, because dynamic errors are not allowed to disappear when we provide type annotations.

This scenario suggests two possibilities for repairing the DGG: dropping the NSU discipline in favor of type-guided classification, or vice versa. The first option is problematic because it is hard to find other ways of enforcing noninterference. One possibility would be to modify the semantics of conditionals so that they raise the secrecy of all references that could be updated in either branch [24]. In Figure 1, this would mean raising y's label above x's even when the else branch is taken. Apart from the potential performance impact, implementing this solution in any realistic language would require a rich analysis to compute write sets, which would likely push us further towards a static type system. And even if we decided that this was worth it, keeping type-guided classification would be problematic for another popular feature of IFC: first-class labels.

Labels are first class if they can be manipulated programmatically; for instance, we might write `labelOf b == S` to test whether b holds a secret. First-class labels are often adopted in practically minded IFC systems [30, 35] because they enable rich data-dependent policies. Unfortunately, they can easily break the DGG with type-guided classification. Consider Figure 2, for instance: if the DGG were true, the unannotated program would behave the same way as the two annotated ones, which is impossible because they return different results. Similar issues have been observed in languages with dynamic type tests [8, 27]: if programs can test anything about a value's type, they can discern between different static annotations.

Thus, to reconcile noninterference and gradual typing, we are led to the second option: abandoning type-guided classification. The effect of an annotation should be merely to check labels, not to modify them. For Figure 1, this would mean that b, y and z would still be dynamically labeled P despite the static annotation Bool<S>, triggering an NSU error without any harm to the gradual guarantee. Likewise, the annotations in Figure 2 could lead to a cast error, but they would not change the result of the test. Modifying labels should still be possible, but through a *term-level* operation that is not covered by the DGG.

We realize this idea with *GLIO*, a gradual language based on the LIO library [31]. LIO exposes an API for securely manipulating secret data, to which GLIO adds optional annotations for preventing security errors statically. Following the tradition of gradual typing, GLIO features a notion of *consistent subtyping* to allow annotated and unannotated code to interoperate automatically, unlike prior work [9], where

```
let b : Bool<S> = true in labelOf b == S
let b : Bool<P> = true in labelOf b == S
let b = true in labelOf b == S
```

**Figure 2.** Failure of the DGG with first-class labels and type-guided classification. The first two programs have no reason to fail, and with type-guided classification they terminate successfully with different results. The DGG would force the third program must behave the same way as the first two, which is impossible.

annotations might need to be checked manually. We still need to investigate if GLIO could be embedded in Haskell like LIO, but a standalone implementation should pose no challenges.

An important characteristic of gradual type systems is how much support they provide for transitioning legacy programs to richer type disciplines. The literature on gradual IFC offers different answers to this question; ML-GS [11], for example, requires references to be given an explicit secrecy label, and thus does not directly apply to legacy programs, while $GSL_{Ref}$ [32] allows omitting all such annotations. By extending LIO, GLIO adopts a mixed stance in that regard. On the one hand, LIO does require programs to provide term-level annotations for certain operations, including reference allocation. On the other hand, LIO's *coarse-grained design* obviates the need for tracking labels in most of the program; most values are protected by the *PC label*, a state component used in NSU checks.

In principle, it would be possible to allow missing label annotations for references in GLIO by choosing a default value for them, such as the current PC label. Unfortunately, the benefits of this approach would be limited for gradual typing: in the presence of first-class labels, no analog of the DGG can hold when overriding these defaults. We do not know if the situation fundamentally changes if first-class labels are absent, but missing reference annotations are not the only source of violations for the DGG: similar issues arise in $GSL_{Ref}$ even if all reference annotations are present, by adapting the counterexample of Figure 1 to its syntax.

*Our contributions*, in sum, are as follows. We introduce GLIO, a gradual language based on LIO with higher-order functions and storage, flow-insensitive references, coarse-grained IFC, security subtyping and public, first-class labels. After an informal tour of the language in Section 2, we present its syntax and type system in Section 3, and define its semantics in Section 4. We prove that GLIO satisfies both termination- and error-insensitive noninterference (Section 5) and the gradual guarantee of Siek et al. [27] (Section 6). We discuss related work in Section 7 and conclude in Section 8. Detailed proofs and definitions are included in Appendix A.

## 2 Overview

Before diving into technical details, we give a brief tour of GLIO. Traditionally, IFC languages have followed a *fine-grained* discipline: every value carries a secrecy label, which is implicitly checked and propagated on every operation (statically or dynamically). This category includes $\lambda^{info}$ [4], Flow Caml [22] and Jif [18], among others. By contrast, systems such as DCC [1], LIO [31] and GLIO follow a *coarse-grained* discipline: only certain values carry labels, and they must be manipulated using special primitives. The two styles are equally expressive [23, 34], but coarse-grained systems are easier to implement (since they track less information) and offer a smoother migration path to legacy programs (since most of the code does not need to worry about IFC).

Following LIO, GLIO places labeled values in a special type called `Lab`, and uses a monad `LIO` to express computations that handle secrets. Its most basic primitives are:

```
label    :: Label -> a -> LIO (Lab a)
unlabel  :: Lab a -> LIO a
labLabel :: Lab a -> Label
pcLabel  :: LIO Label
```

The types shown here mimic those of the original LIO, but we'll soon see that they can be refined with secrecy annotations. The `label` and `unlabel` functions are used to wrap a value of type `a` with a secrecy label and to unwrap it. To do this safely, the `LIO` monad encapsulates a state component known as the *PC label*, as usual in dynamic IFC. This label bounds the secrecy of all the values that have been unlabeled during the computation. Before assignments, the program performs an NSU check on this label to determine whether the operation is safe. The functions `labLabel` and `pcLabel` allow inspecting the label of a labeled value and the current PC label.

The behavior of these primitives is illustrated in Figure 3, which shows a loose translation of Figure 1 into GLIO. In addition to the explicitly labeled values, the main difference with respect to Figure 1 is the `new` operator, which takes a secrecy label `P` as its argument. This translation is contrived for a coarse-grained system because of the spurious wrapping of the boolean `b`, but it is operationally closer to the original example and gives an idea of how GLIO enforces the DGG.

The program runs the same way as before. Unlabeling `b` amounts to a no-op: since its label is public, we do not need to update the PC label. On the contrary, `x` is marked as secret, so unlabeling it has the effect of bumping the PC label to `S`. This change is detected by GLIO's NSU check, which deems the update to `y` unsafe and halts the program with an error.

Instead of `Lab Bool`, we could have given `b` the more precise type `Lab[S] Bool`, which says that the *dynamic* secrecy of the wrapped boolean is bounded by `S`. Since this label is `P`, which is below `S`, the assignment can be performed safely. Importantly, this does not modify `b`'s label, and updating `y`

```
f :: Lab Bool -> LIO Bool
f x = do
  -- Alternative annotation: Lab[S] Bool
  b :: Lab Bool <- label P True
  b' <- unlabel b
  y  <- new P b'
  z  <- new P b'
  x' <- unlabel x
  if x' then set y False
    else return ()
  y' <- get y
  if y' then set z False
    else return ()
  get z

do { x <- label S True; f x }
```

**Figure 3.** Translation of the example of Figure 1 into GLIO

leads to the same result as before: an error. Since the behavior of the program did not change after refining the type, the DGG has not been violated.

The annotation did not break the DGG, but it was also not strong enough to catch the IFC error statically. Figure 4 demonstrates how this could be done in GLIO with a fully annotated version of the previous program. As in HLIO [9], the annotations on the `LIO` monad provide upper bounds on the PC label at the beginning and at the end of the computation. The annotations on `Ref` are stricter than those for `Lab`: instead of an upper bound, they give the exact secrecy of the contents the reference. This is to ensure safety: if the static label of a reference, `S`, were above its actual dynamic label, say `P`, the NSU check would still throw an error at run time, which the type checker would not be able to prevent.

To check `unlabel`, the type system propagates the static label of its argument into the PC label. Since `x` could be a secret, the type system rejects the assignment to `y`, as it could lead to an illegal implicit flow.

Figure 5 presents a middle ground between dynamic and static enforcement, using label introspection to test whether the NSU check would fail. Unlike labeled values, dynamic labels are themselves *public*, and can be inspected without tainting the PC. The `lub` operator computes the *join*, or least upper bound, of two labels, while `canFlowTo` checks if one label is below another. If the test passes, the assignment is performed without triggering any errors. Otherwise, the program logs the unsafe condition so that more robust recovery code can act later.

***Labeling and allocation.*** Figure 6 further details the role of labels in values and references. The first program, `refLab`, stores the contents of a labeled value `x` in a fresh reference `r`. In this example, the new reference is typed as `Ref[S] Bool`

```
h :: Lab[S] Bool -> LIO[P,S] Bool
h x = do
  -- PC label = P
  b  :: Lab[P] Bool <- label P True
  b' :: Bool        <- unlabel b
  y  :: Ref[P] Bool <- new P b'
  z  :: Ref[P] Bool <- new P b'
  x' :: Bool        <- unlabel x
  -- PC label = S
  if x'
  -- Assignment is rejected
    then set y False
    else return ()
  y <- get y
  if y then set z False
    else return ()
  get z

do { x <- label S True; g x }
```

**Figure 4.** A fully annotated version of Figure 3 that is rejected at compile time

```
maybeUpdate :: Ref Bool -> Lab Bool -> LIO ()
maybeUpdate r x = do
  lpc <- pcLabel
  let lx = labLabel x
  let lr = refLabel r
  if lpc `lub` lx `canFlowTo` lr then do
    x' <- unlabel x
    set r x'
  else set errorOccurred True
```

**Figure 5.** Error prevention through label introspection

```
refLab :: Lab[S] Bool -> LIO[P,P] (Ref[S] Bool)
refLab x = do
  r :: Ref[S] Bool <- new S true
  -- toLab :: Label -> LIO a -> LIO (Lab a)
  toLab S (do { x' <- unlabel x; set r x' })
  return r

labRef :: Ref Bool -> LIO[P,P] (Lab Bool)
labRef r = toLab (refLabel r) (get r)

eqRef :: Ref Bool -> Ref Bool -> LIO[P,P] Bool
eqRef r1 r2 = return (r1 == r2)
```

**Figure 6.** Labeling and dynamic allocation

because the annotation is constant, but in general this argument can be an arbitrarily complex expression, in which case the reference would get the imprecise type `Ref Bool`.

```
labCast :: LIO (Lab[P] Bool)
labCast = do
  b :: Lab[P] Bool <- label P True
  return (b :: Lab[S] Bool :: Lab Bool
            :: Lab[P] Bool)

labClass :: LIO (Lab[P] Bool)
labClass = do
  b :: Lab[P] Bool <- label P True
  b'  <- unlabel b
  b'' <- label S b'
  return (b' :: Lab Bool :: Lab[P] Bool)

refCast :: LIO (Ref[S] Bool)
refCast = do
  r :: Ref[P] Bool <- new P True
  return (r :: Ref Bool :: Ref[S] Bool)
```

**Figure 7.** Casts in GLIO

For the allocation to succeed, the reference label must be above the PC label, which can be statically enforced in this case thanks to the PC annotations.

The function uses another primitive of GLIO, `toLab`, to avoid raising the PC label too much and causing spurious NSU errors—a problem known in the literature as *label creep*. The first argument of `toLab` is a label `l` that bounds the confidentiality of the result,[2] and its second argument is a computation `f`. If the final PC label after running `f` is below `l`, `toLab` wraps the result in a value labeled with `l` and restores the PC label to its original value; otherwise, it throws an error. In `refLab`, the annotations are enough to guarantee the absence of errors and indicate that the PC label is indeed restored at the end of execution.

The second program, `labRef`, goes in the opposite direction: it uses `toLab` to wrap the contents of `r` into a labeled value of the same secrecy as `r`.

Fine-grained IFC often makes a distinction between the label of a reference, which protects its identity, and the label of its contents. In GLIO, what is sometimes called the "label of the reference" refers actually to the label of its contents: the identity of the reference is always public with respect to the PC label, and does not need to be protected with special checks. This is illustrated in the third program, `eqRef`, which tests if two references are identical. This comparison does not take their contents into account, which is why the PC label does not have to be tainted.

***Casts and classification.*** GLIO includes a notion of consistent subtyping to allow annotated and unannotated code

[2] You may wonder why the first argument of `toLab` is needed, since we could have also used the final PC label to wrap the result. The problem is that labels in GLIO are public, and can be used to leak secrets [15]. By fixing the final label from the onset, we avoid the issue.

to interoperate. For example, we may pass a value `r` of type `Lab Bool` to `refLab` in Figure 6, and the language inserts the appropriate dynamic checks to ensure safety. (In this case, the checks are guaranteed to succeed, assuming the argument's static label `S` denotes maximum secrecy.)

We can also trigger casts explicitly using type ascription, as shown in Figure 7. The first function, `labCast`, labels the boolean `True` with `P` and sends it through a series of casts, indicated with the `::` operator. The type system checks each cast to rule out obvious or potential errors, such as coercing `Bool` to `Unit` or `Lab[S] Bool` to `Lab[P] Bool`.

Once `labCast` reaches the last cast to `Lab[P] Bool`, it successfully returns `True` labeled as `P`, because the final label on the boolean stays the same across the casts—in other words, classification and type casts are decoupled. This contrasts with previous work [11, 12], in particular with $\mathrm{GSL}_{\mathsf{Ref}}$ [32], which by design would trigger a run-time error, since it treats the last cast as a declassification. This behavior can be replicated in GLIO by replacing the first cast to `Lab[S] Bool` with another call to `label`, as shown in the second program, `labClass`. Classification succeeds, because `S` is more secret than `P`, but the last cast fails for the same reason.

Finally, `refCast` demonstrates the difference between labels for `Ref` and `Lab`. The annotations on reference types fix the labels of their contents, so the final cast to `Ref[S] Bool` fails during execution even though `S` is more secret than `P`. Note that this cast has to come after a cast to the imprecise type `Ref Bool`: were it omitted, the type checker would reject the program, as such a coercion always fails.

## 3 Language

Having built basic intuition, we are ready for a formal definition. The development assumes a lattice of secrecy labels $l \in L$ ordered by $\preccurlyeq$, comprising joins $\vee$, meets $\wedge$, a bottom element $\bot$ and a top element $\top$. The higher a label, the more confidential the values it classifies, with $\bot$ denoting public data and $\top$ denoting maximum secrecy. A simple choice for $L$ would be a lattice of labels $\{\mathsf{P}, \mathsf{S}\}$ ordered by $\mathsf{P} \preccurlyeq \mathsf{S}$. A more interesting instance is $L = \mathscr{P}(P)$ ordered by the subset relation, where $P = \{\mathsf{Alice}, \mathsf{Bob}, ...\}$ is a set of principals that own data, $\bot = \emptyset$, $\top = P$, $\wedge = \cap$ and $\vee = \cup$.

Figure 8 summarizes the syntax of terms and types. To simplify the development, we modify the informal overview of the previous section in two aspects. First, since our main technical challenges pertain to impure code, we conflate pure functions and the LIO monad into a single type $T \xrightarrow{\bar{l}_1,\bar{l}_2} S$, which intuitively corresponds to the type $T \longrightarrow \mathsf{LIO}[\bar{l}_1, \bar{l}_2]\, S$ seen earlier. Because of cast errors, "pure" code in our language still needs to be managed monadically, and this simplification allows us to model pure and impure code with a single monad (cf. Section 4). Second, to allow for a more compact semantics later, we present the syntax in A-normal form [13, 25]: most term formers only allow variables as arguments, and the earlier snippets should be translated into a sequence of let definitions. The first term rows contain usual constructs for manipulating booleans, functions, and the heap. The last rows are specific to IFC, and correspond to the primitives of LIO [30]. Two syntactic forms, new and toLab, take either variables or label constants as arguments to allow for more precise typing rules, as we'll soon see. Type ascription is syntax sugar defined in terms of let, and label is defined in terms of toLab. (Since we don't use a separate monadic type, label and toLab are actually synonyms.)

As usual in gradual languages, the missing annotations in concrete syntax formally correspond to the *gradual label* $? \in \bar{L} \triangleq L \uplus \{?\}$, which represents a statically unknown label. The language does not include product, sum, and recursive types, but we foresee no difficulties in doing so—for recursive types in particular, GLIO already includes a higher-order store, which forces us to handle similar technical challenges.

Figure 9 presents the type system. The label indices in judgments $\Gamma \vdash_{\bar{l}_1,\bar{l}_2} e : T$ correspond to the static annotations on the LIO monad of Section 2: they constrain the PC label at the beginning and at the end of the execution of $e$. The rules reflect the behavior of the programs described earlier. For example, the variable rule does not change the label annotation because variables are already protected by the current PC label, and thus require no additional tainting. A similar reasoning applies to the introspection primitives refLabel, labLabel and pcLabel.

The rule for let shows how the label indices are threaded through as the computation unfolds. The *consistent subtyping* assumption $T' \preccurlyeq T$ allows weakening security annotations or even omitting them entirely. Its definition, shown in Figure 10, resembles the subtyping discipline of Rajani and Garg [23], but adapted to the gradual setting using the *Abstracting Gradual Typing* (AGT) framework [14]. In AGT, a gradual type $T$ is interpreted as a set $\gamma(T)$ of fully annotated types, where each missing annotation is replaced by all possible completions. For example, $\gamma(\mathsf{Lab}_?(\mathsf{Bool}))$ is $\{\mathsf{Lab}_l(\mathsf{Bool}) \mid l \in L\}$. (Figure 20 gives the complete definition.) This allows us to lift arbitrary predicates on fully annotated types to gradual types: the inductive presentation of Figure 10 is equivalent to saying that $T \preccurlyeq S$ holds precisely when there exist $T' \in \gamma(T)$ and $S' \in \gamma(S)$ such that $T' \preccurlyeq S'$, for a suitable subtyping relation $\preccurlyeq$ on fully annotated types. The $\preccurlyeq$ relation on $\bar{L}$, which extends the one on $L$, can be recast in the same way, by posing $\gamma(?) = L$ and $\gamma(l) = \{l\}$.

On multiple rules, the consistent ordering on gradual labels is used to rule out IFC errors. For example, the side condition on the set rule subsumes the corresponding NSU check. Other rules, such as get and if, taint types and the PC label using *partial* consistent join operations $\curlyvee$ (Figure 11). The definition uses a consistent meet operation $\curlywedge$ and an intersection operation $\cap$ on types and gradual labels. These operations are not joins and meets in the usual sense, since

$$
\begin{aligned}
l &\in L \\
\bar{l} &\in \bar{L} \triangleq L \uplus \{?\} \\
b &\in \{0, 1\} \\
c &\in L \uplus \{x, y, z, \ldots\} \\
\oplus &\in \{\wedge, \vee\} \\
\Gamma &\in \mathsf{Var} \rightharpoonup_{\mathrm{fin}} \mathsf{Type} \\
e :: T &\triangleq \mathsf{let}(e, x : T \ldots) \\
\mathsf{label}(c, x) &\triangleq \mathsf{toLab}(c, x)
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{Term} \ni e ::= {}& x \mid \mathsf{let}(e_1, x : T.e_2) \mid \mathsf{unit} \mid b \mid \mathsf{if}(x, e_1, e_0) \mid \mathsf{fun}(x :_{\bar{l}} T.e) && \text{standard} \\
&\mid \mathsf{app}(x, y) \mid \mathsf{get}(x) \mid \mathsf{set}(x, y) \mid \mathsf{new}(c, y) \mid \mathsf{eqRef}(x, y) \\
&\mid \mathsf{refLabel}(x) \mid \mathsf{labLabel}(x) \mid \mathsf{pcLabel}() && \text{IFC specific} \\
&\mid l \mid x \oplus y \mid x \preccurlyeq y \mid \mathsf{unlabel}(x) \mid \mathsf{toLab}(c, e) \\
\mathsf{Type} \ni T, S ::= {}& \mathsf{Unit} \mid \mathsf{Bool} \mid \mathsf{Label} \mid \mathsf{Ref}_{\bar{l}}(T) \mid \mathsf{Lab}_{\bar{l}}(T) \mid T \xrightarrow{\bar{l}_1, \bar{l}_2} S
\end{aligned}
$$

**Figure 8.** Syntax of terms and types

$$
\frac{\Gamma(x) = T}{\Gamma \vdash_{\bar{l},\bar{l}} x : T}
\qquad
\frac{\Gamma \vdash_{\bar{l}_1,\bar{l}_2} e_1 : T' \qquad \Gamma[x \mapsto T] \vdash_{\bar{l}_2,\bar{l}_3} e_2 : S \qquad T' \preccurlyeq T}{\Gamma \vdash_{\bar{l}_1,\bar{l}_3} \mathsf{let}(e_1, x : T.e_2) : S}
\qquad
\frac{}{\Gamma \vdash_{\bar{l},\bar{l}} \mathsf{unit} : \mathsf{Unit}}
\qquad
\frac{}{\Gamma \vdash_{\bar{l},\bar{l}} b : \mathsf{Bool}}
$$

$$
\frac{\Gamma(x) = \mathsf{Bool} \qquad \Gamma \vdash_{\bar{l}_1,\bar{l}_2^1} e_1 : T_1 \qquad \Gamma \vdash_{\bar{l}_1,\bar{l}_2^0} e_0 : T_0}{\Gamma \vdash_{\bar{l}_1,\bar{l}_2^1 \vee \bar{l}_2^0} \mathsf{if}(x, e_1, e_0) : T_1 \vee T_0}
\qquad
\frac{\Gamma(x) = \mathsf{Ref}_{\bar{l}}(T)}{\Gamma \vdash_{\bar{l}_1,\bar{l}_1 \vee \bar{l}} \mathsf{get}(x) : T}
$$

$$
\frac{\Gamma(x) = \mathsf{Ref}_{\bar{l}}(T_1) \qquad \Gamma(y) = T_2 \qquad T_2 \preccurlyeq T_1 \qquad \bar{l}_1 \preccurlyeq \bar{l}}{\Gamma \vdash_{\bar{l}_1,\bar{l}_1} \mathsf{set}(x, y) : \mathsf{Unit}}
\qquad
\frac{\Gamma(y) = T \qquad \bar{l}_2 \preccurlyeq l_1}{\Gamma \vdash_{\bar{l}_2,\bar{l}_2} \mathsf{new}(l_1, y) : \mathsf{Ref}_{l_1}(T)}
\qquad
\frac{\Gamma(x) = \mathsf{Label} \qquad \Gamma(y) = T}{\Gamma \vdash_{\bar{l}_2,\bar{l}_2} \mathsf{new}(x, y) : \mathsf{Ref}_{?}(T)}
$$

$$
\frac{\Gamma(x) = \mathsf{Ref}_{\bar{l}_1}(T_1) \qquad \Gamma(y) = \mathsf{Ref}_{\bar{l}_2}(T_2) \qquad T_1 \preccurlyeq T_2 \qquad T_2 \preccurlyeq T_1}{\Gamma \vdash_{\bar{l},\bar{l}} \mathsf{eqRef}(x, y) : \mathsf{Bool}}
\qquad
\frac{\Gamma[x \mapsto T] \vdash_{\bar{l}_1,\bar{l}_2} e : S}{\Gamma \vdash_{\bar{l},\bar{l}} \mathsf{fun}(x :_{\bar{l}_1} T.e) : T \xrightarrow{\bar{l}_1,\bar{l}_2} S}
$$

$$
\frac{\Gamma(f) = T_1 \xrightarrow{\bar{l}_2,\bar{l}_3} S \qquad \Gamma(x) = T_2 \qquad T_2 \preccurlyeq T_1 \qquad \bar{l}_1 \preccurlyeq \bar{l}_2}{\Gamma \vdash_{\bar{l}_1,\bar{l}_3} \mathsf{app}(f, x) : S}
\qquad
\frac{\Gamma(x) = \mathsf{Ref}_{\bar{l}}(T)}{\Gamma \vdash_{\bar{l}',\bar{l}'} \mathsf{refLabel}(x) : \mathsf{Label}}
\qquad
\frac{\Gamma(x) = \mathsf{Lab}_{\bar{l}}(T)}{\Gamma \vdash_{\bar{l}',\bar{l}'} \mathsf{labLabel}(x) : \mathsf{Label}}
$$

$$
\frac{}{\Gamma \vdash_{\bar{l},\bar{l}} \mathsf{pcLabel}() : \mathsf{Label}}
\qquad
\frac{}{\Gamma \vdash_{\bar{l},\bar{l}} l : \mathsf{Label}}
\qquad
\frac{\Gamma(x) = \Gamma(y) = \mathsf{Label}}{\Gamma \vdash_{\bar{l},\bar{l}} x \preccurlyeq y : \mathsf{Bool}}
\qquad
\frac{\Gamma(x) = \Gamma(y) = \mathsf{Label}}{\Gamma \vdash_{\bar{l},\bar{l}} x \oplus y : \mathsf{Label}}
\qquad
\frac{\Gamma(x) = \mathsf{Lab}_{\bar{l}'}(T)}{\Gamma \vdash_{\bar{l},\bar{l} \vee \bar{l}'} \mathsf{unlabel}(x) : T}
$$

$$
\frac{\Gamma \vdash_{\bar{l}_2,\bar{l}_3} e : T \qquad \bar{l}_3 \preccurlyeq l_1 \vee \bar{l}_2}{\Gamma \vdash_{\bar{l}_2,\bar{l}_2} \mathsf{toLab}(l_1, e) : \mathsf{Lab}_{l_1}(T)}
\qquad
\frac{\Gamma(x) = \mathsf{Label} \qquad \Gamma \vdash_{\bar{l}_2,\bar{l}_3} e : T}{\Gamma \vdash_{\bar{l}_2,\bar{l}_2} \mathsf{toLab}(x, e) : \mathsf{Lab}_{?}(T)}
$$

**Figure 9.** Typing rules

the consistent orders are not transitive, and thus not actual orders; nevertheless, we can show

$$\bar{l}_1 \wedge \bar{l}_2 \preccurlyeq \bar{l}_i \preccurlyeq \bar{l}_1 \vee \bar{l}_2 \quad \text{and} \quad T_1 \wedge T_2 \preccurlyeq T_i \preccurlyeq T_1 \vee T_2$$

for $i \in \{1, 2\}$, whenever the result of these operations is defined. Note that when all labels are ?, $T \preccurlyeq S$ is equivalent to $T = S$, so consistent joins become trivial and the type system reduces to a simplified version of LIO.

The two variants of new and toLab use different typing rules because the secrecy of their results is determined by their label argument. When this label is statically known (that is, in the $\mathsf{new}(l, -)$ and $\mathsf{toLab}(l, -)$ variants), the type system uses it in the result type. When this label is chosen dynamically, the result type is labeled with ?.

The rule for toLab is slightly more permissive than the corresponding dynamic checks in LIO [31], which would translate as $\bar{l}_3 \preccurlyeq l_1$ instead of $\bar{l}_3 \preccurlyeq l_1 \vee \bar{l}_2$. Intuitively, our variant is sound because the result of toLab is protected by both the ascribed label $l_1$ and the initial PC label $\bar{l}_2$. In Section 4, we will see that toLab takes the PC label into account during execution too.

$$\frac{}{\bar{l} \preccurlyeq ?} \qquad \frac{}{? \preccurlyeq \bar{l}} \qquad \frac{l_1 \preccurlyeq l_2 \;:\; L}{l_1 \preccurlyeq l_2 \;:\; \bar{L}} \qquad \frac{T \in \{\mathsf{Unit}, \mathsf{Bool}, \mathsf{Label}\}}{T \preccurlyeq T}$$

$$\frac{\bar{l}_1 \preccurlyeq \bar{l}_2 \qquad \bar{l}_2 \preccurlyeq \bar{l}_1 \qquad T_1 \preccurlyeq T_2 \qquad T_2 \preccurlyeq T_1}{\mathsf{Ref}_{\bar{l}_1}(T_1) \preccurlyeq \mathsf{Ref}_{\bar{l}_2}(T_2)}$$

$$\frac{\bar{l}_1 \preccurlyeq \bar{l}_2 \qquad T_1 \preccurlyeq T_2}{\mathsf{Lab}_{\bar{l}_1}(T_1) \preccurlyeq \mathsf{Lab}_{\bar{l}_2}(T_2)}$$

$$\frac{\bar{l}_1' \preccurlyeq \bar{l}_1 \qquad \bar{l}_2 \preccurlyeq \bar{l}_2' \qquad T_1' \preccurlyeq T_1 \qquad T_2 \preccurlyeq T_2'}{T_1 \xrightarrow{\bar{l}_1, \bar{l}_2} T_2 \preccurlyeq T_1' \xrightarrow{\bar{l}_1', \bar{l}_2'} T_2'}$$

**Figure 10.** Consistent subtyping

$$\begin{aligned}
\bar{l} \oplus ? = ? \oplus \bar{l} &\triangleq ? \\
\bar{l} \cap ? = ? \cap \bar{l} &\triangleq \bar{l} \\
l_1 \oplus l_2 &\triangleq l_1 \oplus l_2 \\
\bar{l} \cap \bar{l} &\triangleq \bar{l} \\
\mathsf{Unit} \oplus \mathsf{Unit} &\triangleq \mathsf{Unit} \\
\mathsf{Unit} \cap \mathsf{Unit} &\triangleq \mathsf{Unit} \\
\mathsf{Bool} \oplus \mathsf{Bool} &\triangleq \mathsf{Bool} \\
\mathsf{Bool} \cap \mathsf{Bool} &\triangleq \mathsf{Bool} \\
\mathsf{Label} \oplus \mathsf{Label} &\triangleq \mathsf{Label} \\
\mathsf{Label} \cap \mathsf{Label} &\triangleq \mathsf{Label} \\
\mathsf{Ref}_{\bar{l}_1}(T_1) \oplus \mathsf{Ref}_{\bar{l}_2}(T_2) &\triangleq \mathsf{Ref}_{\bar{l}_1 \cap \bar{l}_2}(T_1 \cap T_2) \\
\mathsf{Ref}_{\bar{l}_1}(T_1) \cap \mathsf{Ref}_{\bar{l}_2}(T_2) &\triangleq \mathsf{Ref}_{\bar{l}_1 \cap \bar{l}_2}(T_1 \cap T_2) \\
\mathsf{Lab}_{\bar{l}_1}(T_1) \oplus \mathsf{Lab}_{\bar{l}_2}(T_2) &\triangleq \mathsf{Lab}_{\bar{l}_1 \oplus \bar{l}_2}(T_1 \oplus T_2) \\
\mathsf{Lab}_{\bar{l}_1}(T_1) \cap \mathsf{Lab}_{\bar{l}_2}(T_2) &\triangleq \mathsf{Lab}_{\bar{l}_1 \cap \bar{l}_2}(T_1 \cap T_2)
\end{aligned}$$

$$\begin{aligned}
&T_1 \xrightarrow{\bar{l}_1, \bar{l}_2} T_2 \oplus T_1' \xrightarrow{\bar{l}_1', \bar{l}_2'} T_2' \\
&\triangleq (T_1 \ominus T_1') \xrightarrow{\bar{l}_1 \ominus \bar{l}_1', \bar{l}_2 \oplus \bar{l}_2'} (T_2 \oplus T_2') \\
&\left(T_1 \xrightarrow{\bar{l}_1, \bar{l}_2} T_2\right) \cap \left(S_1 \xrightarrow{\bar{l}_1', \bar{l}_2'} S_2\right) \\
&\triangleq (T_1 \cap S_1) \xrightarrow{\bar{l}_1 \cap \bar{l}_1', \bar{l}_2 \cap \bar{l}_2'} (T_2 \cap S_2)
\end{aligned}$$

**Figure 11.** Gradual meets, gradual joins and intersections for labels and types. Most combinations of types yield undefined results. Here, $\oplus$ stands for either $\vee$ or $\wedge$, and $\ominus$ stands for the other operation.

$$[\![\mathsf{Unit}]\!] \cong 1 \qquad [\![\mathsf{Bool}]\!] \cong 2 \qquad [\![\mathsf{Label}]\!] \cong L$$

$$[\![\mathsf{Ref}_{\bar{l}}(T)]\!] \cong \mathsf{Ref}_{\bar{l}} \qquad [\![\mathsf{Lab}_{\bar{l}}(T)]\!] \cong \mathsf{Lab}_{\bar{l}}([\![T]\!])$$

$$[\![T \xrightarrow{\bar{l}_1, \bar{l}_2} S]\!] \cong [\![T]\!] \xrightarrow{\text{cont}} \mathsf{LIO}_{\bar{l}_1, \bar{l}_2}([\![S]\!])$$

$$\mathsf{Ref}_{\bar{l}} \triangleq \{(r_n, r_{\mathsf{stamp}}, r_{\mathsf{label}}) \in \mathbb{N} \times L \times \gamma(\bar{l}) \mid r_{\mathsf{stamp}} \preccurlyeq r_{\mathsf{label}}\}$$

$$\mathsf{Lab}_{\bar{l}}(X) \triangleq \{x@l \mid x \in X, l \in \downarrow\bar{l}\}$$

$$\begin{aligned}
\mathsf{LIO}_{\bar{l}_1, \bar{l}_2}(X) \triangleq \{f \;:\; &\mathsf{Mem} \times \downarrow\bar{l}_1 \xrightarrow{\text{cont}} \mathsf{Error}(\mathsf{Mem} \times X \times \downarrow\bar{l}_2)_\perp \mid \\
&\forall m_1, l_1, x, m_2, l_2.\ f(m_1, l_1) = (m_2, x, l_2) \Longrightarrow \\
&l_1 \preccurlyeq l_2 \wedge \mathsf{valid}(l_1, m_1, m_2)\}
\end{aligned}$$

$$\downarrow\bar{l} \triangleq \{l' \in L \mid l' \preccurlyeq \bar{l}\} \qquad \mathsf{Error}(X) \triangleq X + \{\mathsf{error}\}$$

$$\mathsf{Mem} \triangleq (T \;:\; \mathsf{Type}^\circ) \times \mathsf{Ref}_? \rightharpoonup_{\text{fin}} [\![T]\!]$$

$$\mathsf{Type}^\circ \triangleq \{T \in \mathsf{Type} \mid T^\circ = T\}$$

$T^\circ \triangleq$
  $T$ with all labels replaced by ? (cf. Figure 21)

$$\begin{aligned}
&\mathsf{valid}(l_1, m_1, m_2) \triangleq \\
&\quad \forall (T, r) \in \mathsf{dom}(m_2). \\
&\qquad l_1 \preccurlyeq r_{\mathsf{label}} \wedge (l_1 \not\preccurlyeq r_{\mathsf{stamp}} \Longrightarrow (T, r) \in \mathsf{dom}(m_1))
\end{aligned}$$

**Figure 12.** Interpretation of types and related constructions on CPOs. To simplify notation, we'll treat the isomorphisms defining $[\![T]\!]$ as equations.

## 4 Semantics

Each type $T$ in GLIO corresponds to a set $[\![T]\!]$ (Figure 12). As the heap can store arbitrary values, $[\![T]\!]$ contains negative recursive occurrences, which requires some care to handle. To solve this issue, we define $[\![T]\!]$ as a CPO rather than a plain set, by solving a domain equation [29]. We briefly review basic notions needed to cover the main contributions, and postpone a detailed description of the construction to Appendix A for the interested readers.

First, by CPO we mean a partially ordered set where all increasing chains $x_0 \sqsubseteq x_1 \sqsubseteq \cdots$ have a least upper bound $\bigsqcup_{i \in \mathbb{N}} x_i$. The notation $X \xrightarrow{\text{cont}} Y$ refers to the CPO of continuous functions between $X$ and $Y$—that is, monotone functions $f \;:\; X \longrightarrow Y$ such that $f\left(\bigsqcup_i x_i\right) = \bigsqcup_i f(x_i)$, ordered pointwise. The *lifted CPO* $X_\perp$ extends the CPO $X$ with a least element $\perp$, which represents nontermination. We use equality, or the *discrete order*, on CPOs such as $\mathsf{Ref}_{\bar{l}}$, Type, $L$ and its subsets. $\mathsf{Error}(X)$ is ordered pointwise. The order $m_1 \sqsubseteq m_2$ on Mem holds when $\mathsf{dom}(m_1) = \mathsf{dom}(m_2)$ and $\forall T, r.\ m_1(T, r) \sqsubseteq m_2(T, r)$.

$$[\![\Gamma \vdash_{\bar{l}_1,\bar{l}_2} e : T]\!] : [\![\Gamma]\!] \xrightarrow{\text{cont}} \mathsf{LIO}_{\bar{l}_1,\bar{l}_2}([\![T]\!]) \qquad [\![\Gamma]\!] \triangleq \prod_{x \in \mathsf{dom}(\Gamma)} [\![\Gamma(x)]\!]$$

$$[\![x]\!](s) \triangleq \mathsf{return}(s(x)) \qquad [\![\mathsf{let}(e_1 : T', x : T.e_2)]\!](s) \triangleq \mathsf{do}\begin{cases} v' \leftarrow [\![e_1]\!](s); \\ v \leftarrow [\![T' \preccurlyeq T]\!](v'); \\ [\![e_2]\!](s[x \mapsto v]) \end{cases} \qquad [\![\mathsf{unit}]\!](s) \triangleq \mathsf{return}(1)$$

$$[\![b]\!](s) \triangleq \mathsf{return}(b) \qquad [\![\Gamma \vdash_{\bar{l}_1,\bar{l}_2^1 \vee \bar{l}_2^0} \mathsf{if}(x, e_1, e_0) : T_1 \vee T_0]\!](s) \triangleq \mathsf{do}\begin{cases} b \triangleq s(x) \\ v \leftarrow [\![e_b]\!](s); \\ [\![\bar{l}_2^b \preccurlyeq \bar{l}_2^1 \vee \bar{l}_2^0]\!]; \\ [\![T_b \preccurlyeq T_1 \vee T_0]\!](v) \end{cases}$$

$$[\![\mathsf{get}(x : \mathsf{Ref}_{\bar{l}}(T))]\!](s) \triangleq \mathsf{do}\begin{cases} v \leftarrow \mathsf{get}_{\_,\_,T}(s(x)); \\ [\![T^\circ \preccurlyeq T]\!](v) \end{cases} \qquad [\![\mathsf{set}(x : \mathsf{Ref}_{\bar{l}}(T_1), y : T_2)]\!](s) \triangleq \mathsf{do}\begin{cases} v \leftarrow [\![T_2 \preccurlyeq T_2^\circ]\!](s(y)); \\ \mathsf{set}_{\_,\_,T_2}(s(x), v') \end{cases}$$

$$[\![\mathsf{new}(l_1, y : T)]\!](s) \triangleq \mathsf{do}\begin{cases} v \leftarrow [\![T \preccurlyeq T^\circ]\!](s(y)); \\ \mathsf{new}_{l_1,\_,T}(l_1, v) \end{cases} \qquad [\![\mathsf{new}(x, y : T)]\!](s) \triangleq \mathsf{do}\begin{cases} v \leftarrow [\![T \preccurlyeq T^\circ]\!](s(y)); \\ \mathsf{new}_{?,\_,T}(s(x), v) \end{cases}$$

$$[\![\mathsf{eqRef}(x, y)]\!](s) \triangleq \mathsf{return}(s(x) = s(y)) \qquad [\![\mathsf{fun}(x :_{\bar{l}_1} T.e)]\!](s) \triangleq \mathsf{return}(\lambda v.[\![e]\!](s[x \mapsto v]))$$

$$[\![\Gamma \vdash_{\bar{l}_1,\bar{l}_3} \mathsf{app}(f : T_1 \xrightarrow{\bar{l}_2,\bar{l}_3} S, x : T_2) : S]\!] \triangleq \mathsf{do}\begin{cases} v \leftarrow [\![T_2 \preccurlyeq T_1]\!](s(x)); \\ [\![\bar{l}_1 \preccurlyeq \bar{l}_2]\!]; \\ s(f)(v) \end{cases} \qquad [\![\mathsf{refLabel}(x)]\!](s) \triangleq \mathsf{return}(s(x)_{\mathsf{label}})$$

$$[\![\mathsf{labLabel}(x)]\!](s) \triangleq \mathsf{do}\begin{cases} \_@l \triangleq s(x); \\ \mathsf{return}(l) \end{cases} \qquad [\![\mathsf{pcLabel}()]\!](s)(m, l) \triangleq (\emptyset, l, l) \qquad [\![l]\!](s) \triangleq \mathsf{return}(l)$$

$$[\![x \preccurlyeq y]\!](s) \triangleq \mathsf{return}(s(x) \preccurlyeq s(y)) \qquad [\![x \oplus y]\!](s) \triangleq \mathsf{return}(s(x) \oplus s(y)) \qquad [\![\mathsf{unlabel}(x)]\!](s) \triangleq \mathsf{unlabel}(s(x))$$

$$[\![\mathsf{toLab}(l_1, e)]\!](s) \triangleq \mathsf{toLab}_{l_1,\_,\_,\_}(l_1, [\![e]\!](s)) \qquad [\![\mathsf{toLab}(x, e)]\!](s) \triangleq \mathsf{toLab}_{?,\_,\_,\_}(s(x), [\![e]\!](s))$$

**Figure 13.** Semantics of typing derivations. The types of some variables and expressions are included for clarity, even though they do not appear in the syntax of terms. Conversely, some of the indices of get, set, new and toLab have been left out, but they can be inferred from the annotations in the corresponding judgments.

Let us explain these definitions before moving on to the semantics of terms. The CPOs $\mathsf{Lab}_{\bar{l}}(X)$ contain elements of $X$ protected by a dynamic label $l$; as explained in Section 2, this label is bounded by the annotation $\bar{l}$, not necessarily equal to it. A reference $r = (r_n, r_{\mathsf{stamp}}, r_{\mathsf{label}})$ carries two labels: $r_{\mathsf{stamp}}$ corresponds to the PC label at the moment of allocation, and $r_{\mathsf{label}}$ corresponds to the secrecy of its contents. As noted in Section 2, $r_{\mathsf{label}}$ must exactly match the static annotation on the reference's type, if one is provided. The stamp is not important for program behavior, but it simplifies the proof of noninterference, for reasons that will soon become clear.

We depart from Haskell by following call-by-value rather than call-by-need: functions take forced values as their arguments, rather than elements of a lifted CPO $X_\perp$. This is merely for organizational purposes: call-by-value allows us to segregate divergence as an effect inside LIO, rather than including it explicitly in the denotation of each type.

The CPO $\mathsf{LIO}_{\bar{l}_1,\bar{l}_2}(X)$ corresponds to the computation types of LIO [30] and HLIO [9]. Its elements are functions that take as inputs a memory (Mem) and a PC label ($\downarrow\bar{l}_1$), and that can either run forever ($\perp$), produce an error, or return *memory updates* (Mem), a result ($X$), and a new PC label ($\downarrow\bar{l}_2$). (Returning updates instead of the final memory is unorthodox, but it simplifies the domain equations, as discussed in Appendix A.2.) The post-condition on the PC label means that it goes up to track inspected secrets. The post-condition valid, explained next, ensures that memory updates do not leak secrets.

A memory $m \in \mathsf{Mem}$ is a function with finite domain that maps a type $T$ and a reference $r$ to a value $v \in [\![T]\!]$. We assume that $T$ has no label annotations, because our semantics doesn't track this information for stored values (we discuss a more efficient approach below). The predicate $\mathsf{valid}(l_1, m_1, m_2)$ describes which memory updates are

$$
\begin{aligned}
\mathsf{unlabel}_{\bar{l}_1,\bar{l}_2,X} &: \mathsf{Lab}_{\bar{l}_2}(X) \longrightarrow \mathsf{LIO}_{\bar{l}_1,\bar{l}_1 \vee \bar{l}_2}(X) \\
\mathsf{unlabel}_{\bar{l}_1,\bar{l}_2,X}(x@l_2)(m, l_1) &\triangleq (\emptyset, x, l_1 \vee l_2) \\
\mathsf{get}_{\bar{l}_1,\bar{l}_2,T} &: \mathsf{Ref}_{\bar{l}_2} \longrightarrow \mathsf{LIO}_{\bar{l}_1,\bar{l}_1 \vee \bar{l}_2}(\llbracket T^\circ \rrbracket) \\
\mathsf{get}_{\bar{l}_1,\bar{l}_2,T}(r)(m, l_1) &\triangleq \begin{cases} (\emptyset, v, l_1 \vee \eta_{\mathsf{label}}) & \text{if } m(T^\circ, r) = v \\ \mathsf{error} & \text{if } (T^\circ, r) \notin \mathsf{dom}(m) \end{cases} \\
\mathsf{set}_{\bar{l}_1,\bar{l}_2,T} &: \mathsf{Ref}_{\bar{l}_1} \times \llbracket T^\circ \rrbracket \longrightarrow \mathsf{LIO}_{\bar{l}_2,\bar{l}_2}(1) \\
\mathsf{set}_{\bar{l}_1,\bar{l}_2,T}(r, v)(m, l_2) &\triangleq \begin{cases} ([T^\circ, r \mapsto v], 1, l_2) & \text{if } l_2 \preccurlyeq \eta_{\mathsf{label}} \text{ and } (T^\circ, r) \in \mathsf{dom}(m) \\ \mathsf{error} & \text{otherwise} \end{cases} \\
\mathsf{new}_{\bar{l}_1,\bar{l}_2,T} &: \gamma(\bar{l}_1) \times \llbracket T^\circ \rrbracket \longrightarrow \mathsf{LIO}_{\bar{l}_2,\bar{l}_2}(\mathsf{Ref}_{\bar{l}_1}) \\
\mathsf{new}_{\bar{l}_1,\bar{l}_2,T}(l_1, v)(m, l_2) &\triangleq \begin{cases} ([r \mapsto v], r, l_2) & \text{if } l_2 \preccurlyeq l_1 \text{ and } r = (T^\circ, (n, l_2, l_1)), \text{ with} \\ & n \triangleq \min\{n \mid (T^\circ, (n, l_2, l_1)) \notin \mathsf{dom}(m)\} \\ \mathsf{error} & \text{otherwise} \end{cases} \\
\mathsf{toLab}_{\bar{l}_1,\bar{l}_2,\bar{l}_3,X} &: \gamma(\bar{l}_1) \times \mathsf{LIO}_{\bar{l}_2,\bar{l}_3}(X) \longrightarrow \mathsf{LIO}_{\bar{l}_2,\bar{l}_2}(\mathsf{Lab}_{\bar{l}_1}(X)) \\
\mathsf{toLab}_{\bar{l}_1,\bar{l}_2,\bar{l}_3,X}(l_1, f)(m, l_2) &\triangleq \begin{cases} (m', v@l_1, l_2) & \text{if } f(m, l_2) = (m', v, l_3) \text{ and } l_3 \preccurlyeq l_1 \vee l_2 \\ \mathsf{error} & \text{if } f(m, l_2) = (m', v, l_3) \text{ and } l_3 \not\preccurlyeq l_1 \vee l_2 \\ & \text{or } f(m, l_2) = \mathsf{error} \\ \bot & \text{if } f(m, l_2) = \bot \end{cases}
\end{aligned}
$$

**Figure 14.** Semantics of typing derivations (continued)

$$
\begin{aligned}
\mathsf{return}_{\bar{l},X} &: X \xrightarrow{\text{cont}} \mathsf{LIO}_{\bar{l},\bar{l}}(X) \\
\mathsf{return}(x)(m, l) &\triangleq (\emptyset, x, l) \\
\mathsf{bind}_{\bar{l}_1,\bar{l}_2,\bar{l}_3,X,Y} &: \mathsf{LIO}_{\bar{l}_1,\bar{l}_2}(X) \times \left(X \xrightarrow{\text{cont}} \mathsf{LIO}_{\bar{l}_2,\bar{l}_3}(Y)\right) \xrightarrow{\text{cont}} \mathsf{LIO}_{\bar{l}_1,\bar{l}_3}(Y) \\
\mathsf{bind}(k, f)(m, l) &\triangleq \begin{cases} (m' \vec{\uplus} m'', y, l'') & \text{if } k(m, l) = (m', x, l') \text{ and} \\ & f(x)(m \vec{\uplus} m', l') = (m'', y, l'') \\ \mathsf{error} & \text{if } k(m, l) = (m', x, l') \text{ and} \\ & f(x)(m \vec{\uplus} m', l') = \mathsf{error} \text{ or } k(m, l) = \mathsf{error} \\ \bot & \text{otherwise} \end{cases} \\
(m \vec{\uplus} m')(r) &\triangleq \begin{cases} m'(r) & \text{if } r \in \mathsf{dom}(m') \\ m(r) & \text{otherwise} \end{cases}
\end{aligned}
$$

**Figure 15.** Monadic operations of LIO

allowed under the PC label $l_1$: new and updated locations must pass the NSU check for $l_1$ ($l_1 \preccurlyeq \eta_{\mathsf{label}}$), and stamps must reflect their allocation context, which, as hinted earlier, is a technical device to simplify the noninterference proof.

The definition of LIO does not preclude computations that access undefined locations in memory, because its elements take all possible memories as their input. It would be possible to rule out these errors with a Kripke semantics in the style of Levy [17], but the issue is orthogonal to our purposes, and we stick to the current formulation for simplicity. Note, however, that some memory-related errors are ruled out by the shape of the memory. For instance, if we try to read $m(\mathsf{Bool}, r)$ and that location is defined, we know that it contains indeed a boolean, which we can access directly.

With the interpretation of types at hand, we are ready for the semantics of typed terms, shown in Figures 13 and 14. We

$$
\begin{aligned}
[\![T \preccurlyeq S]\!]_{\bar{l}} &: [\![T]\!] \xrightarrow{\text{cont}} \mathsf{LIO}_{\bar{l},\bar{l}}([\![S]\!]) \quad (\text{for } T \preccurlyeq S) \\
[\![T \preccurlyeq T]\!] &\triangleq \mathsf{return} \quad (\text{for } T \in \{\mathsf{Unit}, \mathsf{Bool}, \mathsf{Label}\}) \\
[\![\mathsf{Ref}_{\bar{l}_1}(T) \preccurlyeq \mathsf{Ref}_{\bar{l}_2}(S)]\!](n, l_1, l_2) &\triangleq \begin{cases} \mathsf{return}(n, l_1, l_2) & \text{if } l_2 \in \gamma(\bar{l}_2) \\ \lambda(-).\,\mathsf{error} & \text{otherwise} \end{cases} \\
[\![\mathsf{Lab}_{\bar{l}_1}(T_1) \preccurlyeq \mathsf{Lab}_{\bar{l}_2}(T_2)]\!](v@l) &\triangleq \begin{cases} \mathsf{do}\begin{cases} v' \leftarrow [\![T_1 \preccurlyeq T_2]\!](v); \\ \mathsf{return}(v'@l) \end{cases} & \text{if } l \in \downarrow\bar{l}_2 \\ \lambda(-).\,\mathsf{error} & \text{otherwise} \end{cases} \\
[\![T \xrightarrow{\bar{l}_1,\bar{l}_2} S \preccurlyeq T' \xrightarrow{\bar{l}'_1,\bar{l}'_2} S']\!](f) &\triangleq \mathsf{return}\ \lambda x'.\mathsf{do}\begin{cases} x \leftarrow [\![T' \preccurlyeq T]\!](x'); \\ [\![\bar{l}'_1 \preccurlyeq \bar{l}_1]\!]; \\ y \leftarrow f(x); \\ [\![\bar{l}_2 \preccurlyeq \bar{l}'_2]\!]; \\ [\![S \preccurlyeq S']\!](y); \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
[\![\bar{l}_1 \preccurlyeq \bar{l}_2]\!] &: \mathsf{LIO}_{\bar{l}_1,\bar{l}_2}(1) \quad (\text{for } \bar{l}_1 \preccurlyeq \bar{l}_2) \\
[\![\bar{l}_1 \preccurlyeq \bar{l}_2]\!](m, l_1) &\triangleq \begin{cases} (\varnothing, 1, l_1) & \text{if } l_1 \in \downarrow\bar{l}_2 \\ \mathsf{error} & \text{otherwise} \end{cases}
\end{aligned}
$$

**Figure 16.** Label and type coercion

equip LIO with the structure of a parameterized monad [3] (Figure 15), which we use to interpret the Haskell-like do notation in the definitions. Notice how bind applies updates to the initial memory before invoking its continuation, in accordance with our treatment of state. Figure 16 defines the interpretation of subtyping coercions. As explained earlier, coercing a value into Lab or Ref types never changes its label, only checks it, which will be important for the gradual guarantee. Similarly, the coercions triggered by casting or applying a function never modify the PC label.

The behavior of basic ML operations is standard, except for coercions and the NSU checks in set and new. To read a reference, we cast its contents to ensure that the labels on the type are respected; conversely, when updating it reference, we use a cast to forget the labels. (Note that $T \preccurlyeq T^\circ$ and $T^\circ \preccurlyeq T$ hold for every $T$.) A more efficient approach would be to use *monotonic references* [28], whose types are guaranteed to be bounded in precision by the type of their contents during execution. This property ensures that accesses to a reference of fully annotated type can be performed directly, without any casts. We believe that monotonic references could be incorporated in GLIO without compromising our results, but arguing about their correctness requires an intricate stateful invariant, and we keep our scheme for simplicity. Note that in the case of base types, the casts reduce to the identity, because they have no labels to be checked.

The IFC operations are modeled after their analogues in LIO [30], but toLab includes the initial PC label $l_2$ in its side condition, as anticipated by its typing rule. Note how unlabel and get taint the PC label to track the secrecy of the result.

The examples of Section 2 have already exercised the most interesting aspects of the semantics, except for one: stamps. Consider the following program $e$, written in informal syntax for clarity (recall that S stands for ⊤).

```
toLab S $ do
  b' <- unlabel b
  if b' then do { new S True; return () }
    else return () }
new S True
```

We can produce a typing judgment $[b \mapsto \mathsf{Lab}_?(\mathsf{Bool})] \vdash_{\bot,\bot} e : \mathsf{Ref}_\top(\mathsf{Bool})$, which corresponds to a function $[\![e]\!]$ of type $\mathsf{Lab}_?(2) \xrightarrow{\text{cont}} \mathsf{LIO}_{\bot,\bot}(\mathsf{Ref}_\top)$. By running this program on two different inputs and an empty memory, we obtain successful executions

$$
\begin{aligned}
[\![e]\!](1@\top)(\varnothing, \bot) &= ([r_0 \mapsto 1, r_1 \mapsto 1], r_1, \bot) \\
[\![e]\!](0@\top)(\varnothing, \bot) &= ([r_1 \mapsto 1], r_1, \bot),
\end{aligned}
$$

where $r_0 = (\mathsf{Bool}, (0, \top, \top))$ is allocated inside the conditional, and $r_1 = (\mathsf{Bool}, (0, \bot, \top))$ is allocated at the end.

Although the secret $b$ caused $e$ to perform different allocations, the result is the same: the stamps allow us to perform the allocations in high-secrecy contexts without impacting references allocated in low-secrecy contexts. This technique, due to Azevedo de Amorim et al. [5], simplifies the proof of noninterference because we can match references in related executions up to equality. Without stamps, noninterference would still hold, but the values returned in each execution would not necessarily be equal, requiring a more complex argument to relate syntactically different references [7].

| CPO | Relation | Definition |
|---|---|---|
| $1, 2, L, \mathsf{Ref}_{\bar{l}}$ | $x \approx_l y$ | $x = y$ |
| $\mathsf{Lab}_{\bar{l}}(X)$ | $x_1@l_1 \mathrel{\widetilde{\approx}}_l x_2@l_2$ | $\forall i \in \{1,2\}.\ l_i \preccurlyeq l \Longrightarrow (x_1 \approx_l x_2 \wedge l_1 = l_2)$ |
| | $x_1@l_1 \approx_l x_2@l_2$ | $x_1@l_1 \mathrel{\widetilde{\approx}}_l x_2@l_2 \wedge l_1 = l_2$ |
| $X \xrightarrow{\text{cont}} Y$ | $f \approx_l g$ | $\forall x \approx_l y, f(x) \approx_l g(y)$ |
| $\mathsf{LIO}_{\bar{l}_1,\bar{l}_2}(X)$ | $f \approx_l g$ | $\forall m_1 \approx_l m_2, l', m_1', m_2', x_1, x_2, l_1, l_2.$ $f(m_1, l') = (m_1', x_1, l_1) \wedge g(m_2, l') = (m_2', x_2, l_2)$ $\Longrightarrow\ m_1 \vec{\uplus}\ m_1' \approx_l m_2 \vec{\uplus}\ m_2' \wedge x_1@l_1 \mathrel{\widetilde{\approx}}_l x_2@l_2$ |
| $\mathsf{Mem}$ | $m_1 \approx_l m_2$ | $\mathsf{dom}_l(m_1) = \mathsf{dom}_l(m_2) \wedge$ $\forall (T, r) \in \mathsf{dom}(m_1) \cap \mathsf{dom}(m_2).\ m_1(T,r)@r_{\mathsf{label}} \approx_l m_2(T,r)@r_{\mathsf{label}}$ |
| $[\![\Gamma]\!]$ | $s_1 \approx_l s_2$ | $\forall x \in \mathsf{dom}(\Gamma).\ s_1(x) \approx_l s_2(x)$ |

$$\mathsf{dom}_l(m) \triangleq \{(T, r) \in \mathsf{dom}(m) \mid r_{\mathsf{stamp}} \preccurlyeq l\}$$

**Figure 17.** Notions of indistinguishability on CPOs. The definitions assume that the CPOs $X$ and $Y$ carry such notions as well.

## 5 Noninterference

With the semantics pinned down, we are ready for our first main result: showing that GLIO satisfies termination- and error-insensitive noninterference. Informally, an attacker cannot tell the difference between two successful runs of a program that differ only on their secret inputs. To formalize this claim, we follow Abadi et al.'s work on DCC [1] and define a family of relations $(\approx_l)_{l \in L}$ that characterize what elements of $[\![T]\!]$ are indistinguishable to an observer bounded by $l$ (Figure 17).[3] The definition is again circular, but it can be solved with Pitts' framework of relational structures [6, 21], as explained in Appendix A.3.

For base types and references, being indistinguishable simply means being equal. There are two notions of indistinguishability for $\mathsf{Lab}_{\bar{l}}(X)$: weak ($\mathrel{\widetilde{\approx}}_l$) and strong ($\approx_l$). Weak indistinguishability is only an auxiliary notion used to define indistinguishability for computations ($\mathsf{LIO}_{\bar{l}_1,\bar{l}_2}(X)$). We use two notions because GLIO guarantees that the label of a labeled value reveals nothing about the value, whereas the PC label at the end of a computation might reveal something about its result. An observer bounded by $l$ can distinguish two memories if they differ either in their sets of low-stamp locations, $\mathsf{dom}_l$, or in two values stored at a low location.

Our goal is to prove $[\![e]\!] \approx_l [\![e]\!]$ for every well-typed program $e$. This implies that programs do not leak secrets; for example, if $l = \bot$ and $e : \mathsf{Lab}_?(\mathsf{Bool}) \xrightarrow{\bot,\bot} \mathsf{Bool}$, we find that $[\![e]\!](1@\top)(\varnothing, \bot)$ and $[\![e]\!](0@\top)(\varnothing, \bot)$ output the same boolean if both terminate successfully.

**Theorem 5.1** (Noninterference)**.** *If $\Gamma \vdash_{\bar{l}_1,\bar{l}_2} e : T$, we have*

$$[\![e]\!] \approx_l [\![e]\!] : [\![\Gamma]\!] \xrightarrow{\text{cont}} \mathsf{LIO}_{\bar{l}_1,\bar{l}_2}([\![T]\!]).$$

[3] It would be natural to expect indistinguishability to be decreasing with respect to $l$: the more power the attacker has, the more can be distinguished. However, this property is not required to prove noninterference, as evidenced by similar proofs in the literature [1, 23].

*Sketch.* By induction on the typing derivation of $e$. The semantics of the language is defined by using the monadic interface of Figure 15 to compose the operations in Figures 14 and 16. Thus, we just have to show that indistinguishability is preserved by these operations and under composition. The details are discussed in Appendix A.3. ☐

## 6 Gradual guarantees

The main novelty of GLIO is that it satisfies the *dynamic gradual guarantee* [27]: making label annotations more precise can only introduce dynamic type errors, without otherwise changing the behavior of the program.

**Theorem 6.1** (Dynamic Gradual Guarantee, Simple)**.** *Suppose that $e \sqsubseteq e'$ with $\vdash_{\bot,\bar{l}_2} e : T$ and $\vdash_{\bot,\bar{l}_2} e' : T'$.*

- *If $[\![e]\!](\varnothing)(\varnothing, \bot) = \bot$, then $[\![e']\!](\varnothing)(\varnothing, \bot) = \bot$.*
- *If $[\![e]\!](\varnothing)(\varnothing, \bot) = (m, v, l)$, then there exist $m'$ and $v'$ such that $[\![e']\!](\varnothing)(\varnothing, \bot) = (m', v', l)$.*

The premise $e \sqsubseteq e'$, defined on Figure 18, says that $e'$ is obtained from $e$ by replacing some labels on type annotations with ?. The conclusion says that $e$ and $e'$ must behave similarly, except when $e$ throws an error, in which case $e'$ can do whatever it wants. In particular, $e'$ can only fail if $e$ does.

GLIO also satisfies the *static* gradual guarantee, which says that removing label annotations from a term does not break type checking.

**Theorem 6.2** (Static Gradual Guarantee)**.** *If $\Gamma \sqsubseteq \Gamma'$, $\bar{l}_1 \sqsubseteq \bar{l}_1'$, $e \sqsubseteq e'$, and $\Gamma \vdash_{\bar{l}_1,\bar{l}_2} e : T$, there exist $\bar{l}_2' \sqsupseteq \bar{l}_2$ and $T' \sqsupseteq T$ such that $\Gamma' \vdash_{\bar{l}_1',\bar{l}_2'} e' : T'$.*

The proof of this result is a straightforward induction on the typing derivation. Theorem 6.1, on the other hand, requires more care, as the statement is not strong enough to be established directly by induction. We use a generalization similar to prior formulations of the DGG [19, 20].

Labels

$$\frac{l \in L}{l \triangleleft ?} \qquad \frac{l \in L}{l \triangleleft l}$$

Types

$$\frac{T \in \{\mathsf{Unit}, \mathsf{Bool}, \mathsf{Label}\}}{T \triangleleft T} \qquad \frac{\bar{l}_1 \triangleleft \bar{l}_2 \qquad T_1 \triangleleft T_2}{\mathsf{Ref}_{\bar{l}_1}(T_1) \triangleleft \mathsf{Ref}_{\bar{l}_2}(T_2)}$$

$$\frac{\bar{l}_1 \triangleleft \bar{l}_2 \qquad T_1 \triangleleft T_2}{\mathsf{Lab}_{\bar{l}_1}(T_1) \triangleleft \mathsf{Lab}_{\bar{l}_2}(T_2)}$$

$$\frac{\bar{l}_1 \triangleleft \bar{l}'_1 \qquad \bar{l}_2 \triangleleft \bar{l}'_2 \qquad T_1 \triangleleft S_1 \qquad T_2 \triangleleft S_2}{T_1 \xrightarrow{\bar{l}_1,\bar{l}_2} T_2 \triangleleft S_1 \xrightarrow{\bar{l}'_1,\bar{l}'_2} S_2}$$

Environments

$$\frac{\mathsf{dom}(\Gamma_1) = \mathsf{dom}(\Gamma_2) \qquad \forall x.\, \Gamma_1(x) \triangleleft \Gamma_2(x)}{\Gamma_1 \triangleleft \Gamma_2}$$

Terms

$$\frac{}{e \triangleleft e} \qquad \frac{e_1 \triangleleft e'_1 \qquad T \triangleleft T' \qquad e_2 \triangleleft e'_2}{\mathsf{let}(e_1, x : T.\, e_2) \triangleleft \mathsf{let}(e'_1, x : T'.\, e'_2)}$$

$$\frac{e_1 \triangleleft e'_1 \qquad e_2 \triangleleft e'_2}{\mathsf{if}(x, e_1, e_2) \triangleleft \mathsf{if}(x, e'_1, e'_2)}$$

$$\frac{\bar{l} \triangleleft \bar{l}' \qquad T \triangleleft T' \qquad e \triangleleft e'}{\mathsf{fun}(x :_{\bar{l}} T.\, e) \triangleleft \mathsf{fun}(x :_{\bar{l}'} T'\, e')}$$

$$\frac{e \triangleleft e'}{\mathsf{toLab}(l, e) \triangleleft \mathsf{toLab}(l, e')} \qquad \frac{e \triangleleft e'}{\mathsf{toLab}(x, e) \triangleleft \mathsf{toLab}(x, e')}$$

**Figure 18.** Syntactic dynamism relations

**Theorem 6.3** (Dynamic Gradual Guarantee, General)**.** *If* $\Gamma \vdash_{\bar{l}_1,\bar{l}_2} e : T$, $\Gamma' \vdash_{\bar{l}'_1,\bar{l}'_2} e' : T'$, $\Gamma \triangleleft \Gamma'$, $\bar{l}_i \triangleleft \bar{l}'_i$ ($\forall i \in \{1,2\}$), $e \triangleleft e'$ *and* $T \triangleleft T'$, *then* $[\![e]\!] \triangleleft [\![e']\!] : [\![\Gamma]\!] \xrightarrow{\text{cont}} \mathsf{LIO}_{\bar{l}_1,\bar{l}_2}([\![T]\!]) \triangleleft [\![\Gamma']\!] \xrightarrow{\text{cont}} \mathsf{LIO}_{\bar{l}'_1,\bar{l}'_2}([\![T']\!])$.

The *error approximation relations* $[\![e]\!] \triangleleft [\![e']\!]$ in the conclusion are defined on Figure 19. Like indistinguishability in Section 5, they are constructed using Pitts' work [6, 21] (cf. Appendix A.4). A technical subtlety is that the relations are *heterogeneous*: loosening a type $T$ in a term to $S$ requires relating of elements of $[\![T]\!]$ and $[\![S]\!]$. Most clauses of the definition simply lift error approximation pointwise, except for LIO, which exhibits the same asymmetry between $e$ and $e'$ in Theorem 6.1.

The proof of Theorem 6.3, detailed in Appendix A.4, follows the same strategy used for noninterference: we show that the various operations in the semantics preserve $\triangleleft$, and then argue by composition. This is where it is important to ensure that casts do not modify labels: to prove the correctness of operations with casts, we must ensure that $[\![T \preccurlyeq S]\!] \triangleleft [\![T' \preccurlyeq S']\!]$ when $T \triangleleft T'$ and $S \triangleleft S'$. If the choice of $S$ or $S'$ had an impact on labels in the results, these two functions could not be related.

## 7 Related work

***Gradual Typing and IFC.*** One of our main inspirations comes from $\mathrm{GSL}_{\mathsf{Ref}}$ [32], a gradual language for fine-grained IFC. $\mathrm{GSL}_{\mathsf{Ref}}$ suggests an intriguing tension between gradual typing and noninterference. In principle, it could have validated the dynamic gradual guarantee by construction, as it is derived from the AGT methodology [32]. However, a direct application of AGT violated noninterference, just like the example in Figure 1 does if we remove the NSU check from $\lambda^{info}$. The solution of $\mathrm{GSL}_{\mathsf{Ref}}$, unfortunately, was to include an analog of the NSU check that breaks the dynamic gradual guarantee. As hinted in the Introduction, we can witness this failure by adapting the example in Figure 1. The reasons, however, differ slightly from what we've seen earlier.

Unlike most dynamic IFC systems, $\mathrm{GSL}_{\mathsf{Ref}}$ does not describe run-time secrecy with single labels, but with *intervals* of plausible labels. As the program runs, these intervals are refined to rule out labels that invalidate security checks; if they become empty, an error is signaled. This representation, inherited from AGT, allows omitting label annotations entirely from terms and types—a convenient feature for retrofitting IFC to existing programs. Because of the intervals, the checks used by $\mathrm{GSL}_{\mathsf{Ref}}$ to enforce noninterference are more complex than the classic NSU; nevertheless, the gradual guarantee still breaks in the program of Figure 1, because the cast induced by the annotation on `b` ends up modifying the intervals tracked by the program, and thus the result of the NSU analogue.

Rather than adopting $\mathrm{GSL}_{\mathsf{Ref}}$ intervals, GLIO resorts to classic IFC labels and NSU checks. We believe that this choice simplifies the use of first-class labels in a gradual setting, as it is unclear what the semantics of a test `labelOf b == S` should be if `labelOf b` returns a set of plausible labels rather than a single label—for instance, the gradual guarantee would force this result to be consistent for all possible program annotations. Moreover, we can recover some of the benefits of label intervals because most values are unlabeled in our coarse-grained discipline, and because we could easily use a default label when allocating references (e.g. the PC label).

As far as we know, $\mathrm{GSL}_{\mathsf{Ref}}$ was the first work to consider the dynamic gradual guarantee for an IFC language. ML-GS [11] is an earlier design that predates the guarantee, which it can violate by rewriting the program of Figure 1 to classify data through type casts. Other languages use different interpretations of gradual typing from the one adopted

| CPOs | Relation | Definition |
|---|---|---|
| $1, 2, L, \mathsf{Ref}_{\bar{l}}$ | $x \lhd y$ | $x = y$ |
| $\mathsf{Lab}_{\bar{l}}(X)$ | $x_1 @ l_1 \lhd x_2 @ l_2$ | $x_1 \lhd x_2 \wedge l_1 = l_2$ |
| $X \xrightarrow{\text{cont}} Y$ | $f \lhd g$ | $\forall x \lhd y.\ f(x) \lhd g(y)$ |
| $\mathsf{LIO}_{\bar{l}_1,\bar{l}_2}(X)$ | $f \lhd g$ | $\forall m_1 \lhd m_2, l.\ (f(m_1, l) = \bot \implies g(m_2, l) = \bot) \wedge$ <br> $\forall m_1', x_1', l'.\ f(m_1, l) = (m_1', x_1, l')$ <br> $\implies \exists m_2', x_2.\ g(m_2, l) = (m_2', x_2, l') \wedge m_1' \lhd m_2' \wedge x_1 \lhd x_2$ |
| $\mathsf{Mem}$ | $m_1 \lhd m_2$ | $\mathsf{dom}(m_1) = \mathsf{dom}(m_2) \wedge \forall r \in \mathsf{dom}(m_1).\ m_1(r) \lhd m_2(r)$ |
| $[\![\Gamma]\!]$ | $s_1 \lhd s_2$ | $\forall x \in \mathsf{dom}(\Gamma), s_1(x) \lhd s_2(x)$ |

**Figure 19.** Error approximation on CPOs. The relations are heterogeneous, and the left column should be formally understood as describing pairs of CPOs (e.g. the second row defines a relation $(\lhd_{\bar{l},\bar{l}',X,X'}) \subseteq \mathsf{Lab}_{\bar{l}}(X) \times \mathsf{Lab}_{\bar{l}'}(X')$ in terms of another relation $(\lhd_{X,X'}) \subseteq X \times X'$). We will write $x \lhd y : X \lhd Y$ to indicate the CPOs involved in the relation.

here (which goes back to the criteria of Siek et al. [27]), making it hard to provide analogues of the gradual guarantee, because removing annotations might require adding casts to please the type checker. This behavior appears in the language of Disney and Flanagan [10], which interprets missing labels in types as maximum secrecy, and in LJGS [12].

***Dependent Types and IFC.*** Moving further away from gradual typing, we find designs that use dependent types to make static IFC more flexible, deferring label checks to execution time. This category includes the HLIO Haskell library [9] and Jif [18, 35]. Instead of making the checking of dynamic security levels automatic and guided by the structure of types, these systems require programmers to manually check the safety of operations that involve dynamic labels. Thanks to first-class labels, our language allows programmers to perform these tests manually, as in the `maybeUpdate` function in Figure 5. However, because of the lack of dependent types, our type system cannot use the information learned from these tests to rule out errors statically. Bridging the gap between these two kinds of analyses is an interesting avenue for future work.

***Gradual Types and Parametricity.*** Until recently, the interaction between polymorphism and gradual typing exhibited problems similar to the ones we saw for IFC: there had been several proposals of languages that combine the two features [2, 16, 33], but none of them were able to establish both the dynamic gradual guarantee and parametricity. Indeed, Toro et al. [33] conjectured both properties to be fundamentally incompatible.

To solve this issue, New et al. [20] proposed PolyG$^\nu$, a polymorphic calculus based on *term-level sealing*. In PolyG$^\nu$, if we instantiate a polymorphic term $e : \forall^\nu X.\ X \longrightarrow X$ with $\mathsf{Int}$, the result is not of type $\mathsf{Int} \longrightarrow \mathsf{Int}$, but rather of type $X \longrightarrow X$, where $X$ is a *fresh sealed type* generated during execution. To actually use the instantiated function, the sealed type $X$ comes with two conversion functions $\mathsf{seal}_X : \mathsf{Int} \longrightarrow X$ and $\mathsf{unseal}_X : X \longrightarrow \mathsf{Int}$; thus, instead of $e\,[\mathsf{Int}]\ 1 + 1$, as we would write in System F, we would have to write

$$\mathsf{unseal}_X(e\{X \cong \mathsf{Int}\}(\mathsf{seal}_X\, 1)) + 1$$

for the program to be accepted. PolyG$^\nu$ satisfies both the DGG and parametricity; crucially, its DGG does not apply to programs that remove occurrences of seal and unseal, since those live at the term level. Our abandon of type-guided classification is similar: run-time labels are chosen at the term level, and modifying them falls out of the scope of the DGG. This suggests that future tensions with the DGG might be handled by performing at the term level decisions that in fully static systems are usually left implicit at the type level.

## 8 Conclusion

We presented GLIO, a gradual IFC type system based on the LIO library [30] that features higher-order functions, general references, coarse-grained IFC, security subtyping and first-class labels. In addition to noninterference, our type system validates the *dynamic gradual guarantee*, an important correctness criterion for gradual typing. To avoid pitfalls encountered in previous work, we decoupled type annotations from data classification, which our language expresses with typical operations from coarse-grained dynamic IFC.

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# A Detailed proofs and definitions

We detail here definitions and proofs related to the type system, the semantics of types (Section 4), noninterference (Section 5) and the dynamic gradual guarantee (Section 6).

## A.1 Auxiliary Definitions

Figure 20 defines the concretization functions $\gamma$ that interpret gradual types as sets of annotated types. To define the memory in GLIO, we use a type erasure operation $T^{\circ}$, which

$$
\begin{aligned}
\gamma &: \bar{L} \longrightarrow \mathscr{P}(L) \\
\gamma(l) &\triangleq \{l\} \\
\gamma(?) &\triangleq L
\end{aligned}
$$

$$
\begin{aligned}
\gamma &: \mathsf{Type} \longrightarrow \mathscr{P}(\mathsf{Type}) \\
\gamma(\mathsf{Unit}) &\triangleq \{\mathsf{Unit}\} \\
\gamma(\mathsf{Bool}) &\triangleq \{\mathsf{Bool}\} \\
\gamma(\mathsf{Label}) &\triangleq \{\mathsf{Label}\} \\
\gamma(\mathsf{Ref}_{\bar{l}}(T)) &\triangleq \{\mathsf{Ref}_{l}(T') \mid l \in \gamma(\bar{l}), T' \in \gamma(T)\} \\
\gamma(\mathsf{Lab}_{\bar{l}}(T)) &\triangleq \{\mathsf{Lab}_{l}(T') \mid l \in \gamma(\bar{l}), T' \in \gamma(T)\} \\
\gamma\left(T \xrightarrow{\bar{l}_1,\bar{l}_2} S\right) &\triangleq \{T' \xrightarrow{l_1,l_2} S' \mid \\
&\qquad T' \in \gamma(T), l_1 \in \gamma(\bar{l}_1), \\
&\qquad l_2 \in \gamma(\bar{l}_2), S' \in \gamma(S)\}
\end{aligned}
$$

**Figure 20.** Concretization for gradual labels and types

$$
\begin{aligned}
(-)^{\circ} &: \mathsf{Type} \longrightarrow \mathsf{Type} \\
\mathsf{Unit}^{\circ} &\triangleq \mathsf{Unit} \\
\mathsf{Bool}^{\circ} &\triangleq \mathsf{Bool} \\
\mathsf{Label}^{\circ} &\triangleq \mathsf{Label} \\
\mathsf{Ref}_{\bar{l}}(T)^{\circ} &\triangleq \mathsf{Ref}_{?}(T^{\circ}) \\
\mathsf{Lab}_{\bar{l}}(T)^{\circ} &\triangleq \mathsf{Lab}_{?}(T^{\circ}) \\
(T \xrightarrow{\bar{l}_1,\bar{l}_2} S)^{\circ} &\triangleq T^{\circ} \xrightarrow{?,?} S^{\circ}
\end{aligned}
$$

**Figure 21.** Label erasure

replaces all label annotations in $T$ by ?. Its definition is provided in Figure 21. As shown in the following result, erasure, precision and consistent subtyping are naturally related:

**Lemma A.1.** *Label erasure satisfies the following properties for all* $T, S \in \mathsf{Type}$*:*

- $(T^{\circ})^{\circ} = T^{\circ}$.
- *If* $T \sqsubseteq S$*, then* $T \sqsubseteq S^{\circ}$*. In particular,* $S \sqsubseteq S^{\circ}$*.*
- *If* $T^{\circ} \sqsubseteq S^{\circ}$*, then* $T^{\circ} = S^{\circ}$*.*
- *If* $T \lesssim S$*, then* $T^{\circ} = S^{\circ}$*.*

## A.2 Defining the interpretation domains

The CPOs used to interpret types are defined by solving a domain equation with the method of Smyth and Plotkin [29]. We express the domain equation using a mixed-variance functor $F$ on the CPO-category $\mathsf{CPO}_{\perp}^{\mathsf{Type}}$, whose action on objects is shown on Figure 22. Here, $\mathsf{CPO}_{\perp}$ denotes the Kleisli category of the lifting monad $(-)_{\perp}$ on $\mathsf{CPO}$—that is, objects are CPOs, and morphisms are continuous functions of the form $X \xrightarrow{\text{cont}} Y_{\perp}$ equipped with Kleisli composition. We construct an object $D \in \mathsf{CPO}_{\perp}^{\mathsf{Type}}$ equipped with an isomorphism $\mathsf{fold} : F(D, D) \cong D$ by showing that the morphisms of $\mathsf{CPO}_{\perp}^{\mathsf{Type}}$ form a CPO with a least element and that the action of $F$ on morphisms is continuous, among other properties. We define the interpretation of types by setting

$$
\begin{aligned}
[\![T]\!] &\triangleq D(T) \\
\mathsf{LIO}_{\bar{l}_1,\bar{l}_2}(X) &\triangleq \mathsf{LIOF}_{\bar{l}_1,\bar{l}_2}(D, D, X) \\
\mathsf{Mem} &\triangleq \mathsf{MemF}(D),
\end{aligned}
$$

which yields the identities of Figure 12.

You may recall that our semantics treats the result in that computations in LIO as memory updates rather than the final memory. This choice simplifies the definition of LIOF because its action on morphisms is obtained by composing the actions of simpler pieces: $(-) \xrightarrow{\text{cont}} (-)_{\perp}$, MemF, etc. If instead we interpreted the result as being the final memory, we would have to modify LIOF to require that the contents of certain memory locations remain unchanged in the result, which would be inconvenient because the standard action of the functor that maps $(X^{-}, X^{+}, Y)$ to

$$
\mathsf{MemF}(X^{-}) \times \downarrow\bar{l}_1 \xrightarrow{\text{cont}} \mathsf{Error}(\mathsf{MemF}(X^{+}) \times Y \times \downarrow\bar{l}_2)_{\perp}
$$

does not preserve this property.

## A.3 Proving noninterference

To define the indistinguishability relations $(\approx_l)$ of Section 5, we apply the method of Pitts [21], following the formulation of Azevedo de Amorim et al. [6]. We define a $\mathsf{CLat}_{\wedge}$-fibration $q : \mathsf{Ind} \longrightarrow \mathsf{CPO}_{\perp}^{\mathsf{Type}}$ by a change of base:

$$
\begin{tikzcd}
\mathsf{Ind} \arrow[r] \arrow[d, "q"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Adm}^{\mathsf{Type} \times L} \arrow[d, "p^{\mathsf{Type} \times L}"] \\
\mathsf{CPO}_{\perp}^{\mathsf{Type}} \arrow[r, "f"] & \mathsf{CPO}_{\perp}^{\mathsf{Type} \times L}
\end{tikzcd}
$$

where $p : \mathsf{Adm} \longrightarrow \mathsf{CPO}_{\perp}$ is the $\mathsf{CLat}_{\wedge}$-fibration of subsets of CPOs that are closed under limits of chains (so-called *admissible subsets*), and $f(X)(T, l) \triangleq X(T) \times X(T)$. This means that each object of Ind can be seen as a pair $(X, R)$, where $X \in \mathsf{CPO}_{\perp}^{\mathsf{Type}}$ and $(R(T, l) \subseteq X(T) \times X(T))_{T \in \mathsf{Type}, l \in L}$ is a family of relations closed under limits of chains. Moreover, $q$ is admissible, in the sense that Ind is canonically a CPO-category derived from $\mathsf{Adm}^{\mathsf{Type} \times L}$, and this structure is preserved by $q$. We can lift $F$ to a functor $\hat{F}^{\approx}$ on Ind:

$$
\begin{tikzcd}
\mathsf{Ind}^{op} \times \mathsf{Ind} \arrow[r, "\hat{F}^{\approx}"] \arrow[d, "q^{op} \times q"] & \mathsf{Ind} \arrow[d, "q"] \\
\left(\mathsf{CPO}_{\perp}^{\mathsf{Type}}\right)^{op} \times \mathsf{CPO}_{\perp}^{\mathsf{Type}} \arrow[r, "F"] & \mathsf{CPO}_{\perp}^{\mathsf{Type}}
\end{tikzcd}
$$

$$
\begin{aligned}
F : \left(\mathsf{CPO}_\perp^{\mathsf{Type}}\right)^{op} \times \mathsf{CPO}_\perp^{\mathsf{Type}} &\to \mathsf{CPO}_\perp^{\mathsf{Type}} \\
F(X^-, X^+)(\mathsf{Unit}) &\triangleq 1 \\
F(X^-, X^+)(\mathsf{Bool}) &\triangleq 2 \\
F(X^-, X^+)(\mathsf{Label}) &\triangleq L \\
F(X^-, X^+)(\mathsf{Ref}_{\bar{l}}(T)) &\triangleq \mathsf{Ref}_{\bar{l}} \\
F(X^-, X^+)(\mathsf{Lab}_{\bar{l}}(T)) &\triangleq \mathsf{Lab}_{\bar{l}}(X^+(T)) \\
F(X^-, X^+)\left(T \xrightarrow{\bar{l}_1, \bar{l}_2} S\right) &\triangleq X^-(T) \xrightarrow{\text{cont}} \mathsf{LIOF}_{\bar{l}_1, \bar{l}_2}(X^-, X^+, X^+(S))
\end{aligned}
$$

$$
\mathsf{LIOF}_{\bar{l}_1, \bar{l}_2} : \left(\mathsf{CPO}_\perp^{\mathsf{Type}}\right)^{op} \times \mathsf{CPO}_\perp^{\mathsf{Type}} \times \mathsf{CPO}_\perp \to \mathsf{CPO}_\perp
$$

$$
\begin{aligned}
\mathsf{LIOF}_{\bar{l}_1, \bar{l}_2}(X^-, X^+, Y) \triangleq \{f : {} & \mathsf{MemF}(X^-) \times {\downarrow}\bar{l}_1 \xrightarrow{\text{cont}} \mathsf{Error}(\mathsf{MemF}(X^+) \times Y \times {\downarrow}\bar{l}_2)_\perp \mid \\
& \forall m_1, l_1, x, m_2, l_2.\ f(m_1, l_1) = (m_2, x, l_2) \Longrightarrow \\
& l_1 \preccurlyeq l_2 \wedge \mathsf{valid}(l_1, m_1, m_2)\}
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{valid}(l_1, m_1, m_2) \triangleq \\
&\quad \forall (T, r) \in \mathsf{dom}(m_2). \\
&\qquad l_1 \preccurlyeq r_{\mathsf{label}} \wedge (l_1 \not\preccurlyeq r_{\mathsf{stamp}} \Longrightarrow (T, r) \in \mathsf{dom}(m_1))
\end{aligned}
\qquad
\begin{aligned}
\mathsf{MemF} &: \mathsf{CPO}_\perp^{\mathsf{Type}} \to \mathsf{CPO}_\perp \\
\mathsf{MemF}(X) &\triangleq (T : \mathsf{Type}^\circ) \times \mathsf{Ref}_? \rightharpoonup_{\text{fin}} X(T)
\end{aligned}
$$

**Figure 22.** Functors used to interpret types

whose action on objects is given by

$$
\begin{aligned}
&\hat{F}^{\approx}((X^-, R^-), (X^+, R^+))(T, l) \\
&\triangleq (F(X^-, X^+)(T), \hat{F}_R^{\approx}(R^-, R^+)(T, l)),
\end{aligned}
$$

where the relations $\hat{F}_R^{\approx}$ are defined in Figure 23. The existence of this lifting means in particular that $\hat{F}_R^{\approx}$ depends covariantly on $R^+$ and contravariantly on $R^-$ and that it is admissible. Roughly, the last condition holds because the predicates involved in the definition either mention discrete CPOs, for which all sets are admissible, or relations that are assumed to be admissible.

By the aforementioned results, we can construct a family of relations $(R_D^{\approx}(T, l) \subseteq D(T) \times D(T))_{T \in \mathsf{Type}, l \in L}$ such that

$$
\begin{aligned}
&(\mathsf{fold}_T(x), \mathsf{fold}_T(y)) \in R_D^{\approx}(T, l) \\
&\iff (x, y) \in \hat{F}_R^{\approx}(R_D, R_D)(T, l).
\end{aligned}
$$

Given a type $T$ and elements $x, y \in [\![T]\!]$, we pose $x \approx_l y$ if $(x, y) \in R_D^{\approx}(T, l)$, and define similar notations for the auxiliary relations $G^{\approx}$ of Figure 23. Modulo the fold isomorphisms, this mostly matches the definitions given in Figure 17, except for a slight gap between the definition of $\approx_l$ for LIO in Figure 17 and the relations $G_{\mathsf{LIO}}$ of Figure 23: the conclusion of the latter includes the input memories $m_1$ and $m_2$, which are absent in the former. Nevertheless, we can show that the two formulations are equivalent. We begin with the following auxiliary result, which says that indistinguishability is trivial when the initial PC label is high.

**Lemma A.2.** *For all $f, g \in \mathsf{LIO}_{\bar{l}_1, \bar{l}_2}$, suppose that*

$$
f(m_1, l_1) = (m_1', x_1, l_1') \quad \text{and} \quad g(m_2, l_2) = (m_2', x_2, l_2').
$$

*If $m_1 \approx_l m_2$ and $l_i \not\preccurlyeq l$ for all $i$, then $m_1 \vec{\uplus} m_1' \approx_l m_2 \vec{\uplus} m_2'$ and $x_1@l_1' \approx_l x_2@l_2'$.*

*Proof.* Note that $l_i' \not\preccurlyeq l$, since $l_i \preccurlyeq l_i'$ for all $i$ by the definition of LIO. Hence, $x_1@l_1' \approx_l x_2@l_2'$ vacuously. It remains to show that $m_1 \vec{\uplus} m_1' \approx_l m_2 \vec{\uplus} m_2'$.

We first prove that $\mathsf{dom}_l(m_1 \vec{\uplus} m_1') = \mathsf{dom}_l(m_2 \vec{\uplus} m_2')$. Since $\mathsf{dom}_l(m_1) = \mathsf{dom}_l(m_2)$ and for all $i$ we have $\mathsf{dom}_l(m_i \vec{\uplus} m_i') = \mathsf{dom}_l(m_i) \cup \mathsf{dom}_l(m_i')$, it suffices to show

$$
\mathsf{dom}_l(m_i') \subseteq \mathsf{dom}_l(m_i)
$$

for all $i$. If $(T, r) \in \mathsf{dom}(m_i')$ and $r_{\mathsf{stamp}} \preccurlyeq l$, we must have $l_i \not\preccurlyeq r_{\mathsf{stamp}}$, since $l_i \preccurlyeq r_{\mathsf{stamp}}$ would imply the contradiction $l_i \preccurlyeq l$. In this case, the definition of LIO guarantees that $(T, r) \in \mathsf{dom}(m_i)$, and hence $(T, r) \in \mathsf{dom}_l(m_i)$.

To conclude, we must show

$$
(m_1 \vec{\uplus} m_1')(T, r)@r_{\mathsf{label}} \approx_l (m_2 \vec{\uplus} m_2')(T, r)@r_{\mathsf{label}}
$$

for all $T$ and for all $r$, whenever both sides are defined. If $r_{\mathsf{label}} \not\preccurlyeq l$, this is trivial. Otherwise, we must have $(m_i \vec{\uplus} m_i')(T, r) = m_i(T, r)$ for all $i$: if $r_{\mathsf{label}} \preccurlyeq l$ and $r \in \mathsf{dom}(m_i')$, the definition of LIO applied to $f$ implies that $l_1 \preccurlyeq r_{\mathsf{label}}$, hence $l_1 \preccurlyeq l$, a contradiction. We conclude because $m_1 \approx_l m_2$. ∎

$$
\begin{aligned}
\hat{F}^{\approxeq}_R(R^-, R^+)(T, l) &\triangleq \{(x, x) \in X(T) \times X(T)\} \quad (T = \mathsf{Unit}, \mathsf{Bool}, \mathsf{Label}, \mathsf{Ref}_{\bar{l}}(T'))\\
\hat{F}^{\approxeq}_R(R^-, R^+)(\mathsf{Lab}_{\bar{l}}(T), l) &\triangleq \{(x_1@l', x_2@l') \in G^{\approxeq}_{\mathsf{Lab}}(R^+(T, l), l)\}\\
G^{\approxeq}_{\mathsf{Lab}}(R, l) &\triangleq \{(x_1@l_1, x_2@l_2) \mid \forall i \in \{1, 2\}.\ l_i \preccurlyeq l \Longrightarrow ((x_1, x_2) \in R \wedge l_1 = l_2)\}\\
\hat{F}^{\approxeq}_R(R^-, R^+)(T \xrightarrow{\bar{l}_1, \bar{l}_2} S, l) &\triangleq \{(f, g) \mid \forall (x, y) \in R^-(T, l).\ (f(x), g(y)) \in G^{\approxeq}_{\mathsf{LIO}}(R^-, R^+, l)\}\\
G^{\approxeq}_{\mathsf{LIO}}(R^-, R^+, l) &\triangleq \{(f, g) \mid \forall m_1, m_2, l' \preccurlyeq l, m_1', m_2', x_1, x_2, l_1, l_2.\\
&\qquad \mathsf{dom}_l(m_1) = \mathsf{dom}_l(m_2) \wedge (m_1, m_2) \in G^{\approxeq}_{\mathsf{Mem}}(R^-, l)\\
&\qquad \wedge f(m_1, l') = (m_1', x_1, l_1) \wedge g(m_2, l') = (m_2', x_2, l_2) \Longrightarrow\\
&\qquad \mathsf{dom}_l(m_1 \vec{\uplus}\ m_1') = \mathsf{dom}_l(m_2 \vec{\uplus}\ m_2')\\
&\qquad \wedge (m_1', m_2') \in G^{\approxeq}_{\mathsf{Mem}}(R^+, l) \wedge (x_1@l_1, x_2@l_2) \in G^{\approxeq}_{\mathsf{Lab}}(R^+, l)\}\\
G^{\approxeq}_{\mathsf{Mem}}(R, l) &\triangleq \{(m_1, m_2) \mid \forall (T, r) \in \mathsf{dom}(m_1) \cap \mathsf{dom}(m_2).\\
&\qquad (m_1(T, r)@r_{\mathsf{label}}, m_2(T, r)@r_{\mathsf{label}}) \in G^{\approxeq}_{\mathsf{Lab}}(R(T, l), l)\}
\end{aligned}
$$

**Figure 23.** Relational lifting of $F$ for indistinguishability, along auxiliary definitions. Note that the $R$ parameter of $G^{\approxeq}_{\mathsf{Lab}}$ is a single relation, and not a family of relations.

**Lemma A.3.** *Given $f, g \in \mathsf{LIO}_{\bar{l}_1, \bar{l}_2}(X)$, we have $f \approx_l g$ in the sense of Figure 17 if and only if $(f, g) \in G^{\approxeq}_{\mathsf{LIO}}(R^{\approxeq}_D, R^{\approxeq}_D, l)$; that is, when $f$ and $g$ satisfy*

$$
\begin{aligned}
&\forall m_1 \approx_l m_2, l' \preccurlyeq l, m_1', m_2', x_1, x_2, l_1, l_2.\\
&\quad f(m_1, l') = (m_1', x_1, l_1) \wedge g(m_2, l') = (m_2', x_2, l_2) \Longrightarrow\\
&\quad \mathsf{dom}_l(m_1 \vec{\uplus}\ m_1') = \mathsf{dom}_l(m_2 \vec{\uplus}\ m_2')\\
&\quad \wedge (m_1', m_2') \in G^{\approxeq}_{\mathsf{Mem}}(R^{\approxeq}_D, l)\\
&\quad \wedge x_1@l_1 \approxeq_l x_2@l_2.
\end{aligned}
$$

*Proof.* Write $f \equiv_l g$ for $(f, g) \in G^{\approxeq}_{\mathsf{LIO}}(R^{\approxeq}_D, R^{\approxeq}_D, l)$ and $m_1 \equiv_l m_2$ for $(m_1, m_2) \in G^{\approxeq}_{\mathsf{Mem}}(R^{\approxeq}_D, l)$. Note that $m_1 \approx_l m_2 \iff \mathsf{dom}_l(m_1) = \mathsf{dom}_l(m_2) \wedge m_1 \equiv_l m_2$.

($\Longrightarrow$) When all the premises of $f \equiv_l g$ are satisfied, we can apply the hypothesis $f \approx_l g$ to conclude $m_1 \vec{\uplus}\ m_1' \approx_l m_2 \vec{\uplus}\ m_2'$ and $x_1@l_1 \approxeq_l x_2@l_2$. It suffices to show that $m_1' \equiv_l m_2'$. Suppose that we have $(T, r) \in \mathsf{dom}(m_1') \cap \mathsf{dom}(m_2')$. This implies $(T, r) \in \mathsf{dom}(m_1 \vec{\uplus}\ m_1') \cap \mathsf{dom}(m_2 \vec{\uplus}\ m_2')$, which yields, thanks to the above hypothesis,

$$
\begin{aligned}
&m_1'(T, r)@r_{\mathsf{label}}\\
&= (m_1 \vec{\uplus}\ m_1')(T, r)@r_{\mathsf{label}}\\
&\approx_l (m_2 \vec{\uplus}\ m_2')(T, r)@r_{\mathsf{label}}\\
&= m_2'(T, r)@r_{\mathsf{label}}.
\end{aligned}
$$

($\Longleftarrow$) Suppose we have values that satisfy the premises of $f \approx_l g$. There are two cases to consider. If $l' \not\preccurlyeq l$, it suffices to apply Lemma A.2. Otherwise, if $l' \preccurlyeq l$, we can apply the hypothesis $f \equiv_l g$ and conclude $\mathsf{dom}_l(m_1 \vec{\uplus}\ m_1') = \mathsf{dom}_l(m_2 \vec{\uplus}\ m_2')$, $m_1' \equiv_l m_2'$ and $x_1@l_1 \approxeq_l x_2@l_2$. We conclude by noting that $\equiv_l$ is stable under $\vec{\uplus}$, so that $m_1 \vec{\uplus}\ m_1' \equiv_l m_2 \vec{\uplus}\ m_2'$ and thus $m_1 \vec{\uplus}\ m_1 \approx_l m_2 \vec{\uplus}\ m_2'$. □

We are now ready to proceed with the proof of noninterference. We begin with a few auxiliary lemmas about the indistinguishability relation.

**Lemma A.4.** *Let $\equiv_l$ denote one of $\approx_l$ or $\approxeq_l$.*

1. *$x@l' \approx_l y@l' \iff x@l' \approxeq_l y@l'$*
2. *If $x \approx_l y$, then $x@l' \equiv_l y@l'$ for any $l'$.*
3. *If $x@l_x \equiv_l y@l_y$, then $x@(l_x \vee l') \equiv_l y@(l_y \vee l')$.*
4. *If $x@l_x \approxeq_l y@l_y$, then either $l_x = l_y \preccurlyeq l$ and $x \approx_l y$ or $l_x \not\preccurlyeq l$ and $l_y \not\preccurlyeq l$.*

*Proof.*

1. By definition, the first relation is just $x@l' \approxeq_l y@l' \wedge l' = l'$.
2. By the previous item, it suffices to show $x@l' \approxeq_l y@l'$. Unfolding definitions, this means showing that $l' \preccurlyeq l$ implies $x \approx_l y$ and $l' = l'$, which follows from the assumption.
3. First, suppose that $\equiv_l$ is $\approxeq_l$. Assume that one of $l_x \vee l'$ or $l_y \vee l'$ is below $l$. This implies that one of $l_x$ or $l_y$ is below $l$. From the assumption $x@l_x \approxeq_l y@l_y$, we find that $x \approx_l y$ and $l_x = l_y$. We conclude by appealing to the previous item.
   Next, suppose that $\equiv_l$ is $\approx_l$. By definition, we have $x@l_x \approxeq_l y@l_y$ and $l_x = l_y$. The previous sub-proof shows $x@(l_x \vee l') \approxeq_l y@(l_y \vee l')$, which implies $x@(l_x \vee l') \approx_l y@(l_y \vee l')$ by definition.
4. Either $l_x \preccurlyeq l \vee l_y \preccurlyeq l$ or $l_x \not\preccurlyeq l \wedge l_y \not\preccurlyeq l$. In the latter case, we are done. In the former case, the definition of $\approxeq_l$ implies $x \approx_l y$ and $l_x = l_y$, allowing us to conclude.

□

**Lemma A.5.** *Indistinguishability on memories satisfies the following properties.*

1. $\emptyset \approx_l \emptyset\ :\ \mathsf{Mem}$
2. *If $v_1 \approx_l v_2$, then $[T, r \mapsto v_1] \approx_l [T, r \mapsto v_2]$*

3. *If* $m_1 \approx_l m_2$ : Mem *and* $m'_1 \approx_l m'_2$ : Mem, *then* $m_1 \vec{\uplus} m'_1 \approx_l m_2 \vec{\uplus} m'_2$ : Mem

*Proof.* 1. Trivial; the domains are empty, so there are no locations to relate.
2. Since the domains on both sides are equal to $\{(T, r)\}$, we just have to show that $v_1@r_{\mathsf{label}} \approx_l v_2@r_{\mathsf{label}}$. This follows from the assumption and from Lemma A.4.
3. Let $m''_i = m_i \vec{\uplus} m'_i$ for $i \in \{1, 2\}$. We first need to show that the public domains are equal: $\mathsf{dom}_l(m''_1) = \mathsf{dom}_l(m''_2)$. This follows because

$$\begin{aligned} &\mathsf{dom}_l(m''_1)\\ &= \mathsf{dom}_l(m_1) \cup \mathsf{dom}_l(m'_1)\\ &= \mathsf{dom}_l(m_2) \cup \mathsf{dom}_l(m'_2) \qquad \text{(by assumption)}\\ &= \mathsf{dom}_l(m''_2). \end{aligned}$$

Now, suppose that we have some $T$ and some $r$ such that $(T, r) \in \mathsf{dom}(m''_1) \cap \mathsf{dom}(m''_2)$. We need to show

$$m''_1(T, r)@r_{\mathsf{label}} \approx_l m''_2(T, r)@r_{\mathsf{label}}.$$

Thus, assume $r_{\mathsf{label}} \preccurlyeq l$. The definition of references implies that $r_{\mathsf{stamp}} \preccurlyeq r_{\mathsf{label}}$, so $r_{\mathsf{stamp}} \preccurlyeq l$ as well. The hypotheses imply $\mathsf{dom}_l(m_1) = \mathsf{dom}_l(m_2)$ and $\mathsf{dom}_l(m'_1) = \mathsf{dom}_l(m'_2)$, and therefore

$$\begin{aligned} (T, r) \in \mathsf{dom}(m_1) &\iff (T, r) \in \mathsf{dom}(m_2)\\ (T, r) \in \mathsf{dom}(m'_1) &\iff (T, r) \in \mathsf{dom}(m'_2). \end{aligned}$$

Since $(T, r) \in \mathsf{dom}(m''_1)$ and $(T, r) \in \mathsf{dom}(m''_2)$, there are two cases to consider.
- $(T, r) \in \mathsf{dom}(m'_1)$ and $(T, r) \in \mathsf{dom}(m'_2)$. In this case, for every $i \in \{1, 2\}$ we have $m''_i(T, r) = m'_i(T, r)$, and we conclude by using the assumption $m'_1 \approx_l m'_2$.
- $(T, r) \in \mathsf{dom}(m_1)$ and $(T, r) \in \mathsf{dom}(m_2)$ while $(T, r) \notin \mathsf{dom}(m'_1)$ and $(T, r) \notin \mathsf{dom}(m'_2)$. In this case, for every $i \in \{1, 2\}$ we have $m''_i(T, r) = m_i(T, r)$, and we conclude by using the assumption $m_1 \approx_l m_2$.

□

**Lemma A.6.** *We have*
1. $\mathsf{return} \approx_l \mathsf{return} : X \longrightarrow \mathsf{LIO}_{\bar{l},\bar{l}}(X)$
2. $\mathsf{bind} \approx_l \mathsf{bind} : \mathsf{LIO}_{\bar{l}_1,\bar{l}_2}(X) \times (X \longrightarrow \mathsf{LIO}_{\bar{l}_2,\bar{l}_3}(Y)) \longrightarrow \mathsf{LIO}_{\bar{l}_1,\bar{l}_3}(Y)$

*Proof.* For the first point, unfolding definitions, we have to show that, whenever $x_1 \approx_l x_2$, $m_1 \approx_l m_2$ and $l' \in \downarrow\bar{l}$, we have $m_1 \approx_l m_2$ (which was assumed) and $x_1@l' \approxeq_l x_2@l'$ (which follows from Lemma A.4).

For the second point, suppose that we have $f_1 \approx_l f_2 : \mathsf{LIO}_{\bar{l}_1,\bar{l}_2}(X)$ and $g_1 \approx_l g_2 : X \longrightarrow \mathsf{LIO}_{\bar{l}_2,\bar{l}_3}(Y)$. We must show that $\mathsf{bind}(f_1, g_1) \approx_l \mathsf{bind}(f_2, g_2) : \mathsf{LIO}_{\bar{l}_1,\bar{l}_3}(Y)$. Unfolding what this means, suppose that

$$\mathsf{bind}(f_i, g_i)(m_i, l_1) = (m'''_i, v''_i, l''_i) \quad (i \in \{1, 2\}),$$

with $m_1 \approx_l m_2$ and $l_1 \in \downarrow\bar{l}_1$. We have to show that $m'''_1 \approx_l m'''_2$ and $v''_1@l''_1 \approxeq_l v''_2@l''_2$. By the definition of bind, there are $m'_i$, $m''_i$, $v'_i$ and $l'_i$ for $i \in \{1, 2\}$ such that

$$f_i(m_i, l_1) = (m'_i, v'_i, l'_i) \qquad g_i(v'_i)(m_i \vec{\uplus} m'_i, l'_i) = (m''_i, v''_i, l''_i)$$

$$m'''_i = m'_i \vec{\uplus} m''_i.$$

The hypothesis on $f_i$ implies $m_1 \vec{\uplus} m'_1 \approx_l m_2 \vec{\uplus} m'_2$, and $v'_1@l'_1 \approxeq_l v'_2@l'_2$. By Lemma A.4, there are two cases to consider. If $l'_1 = l'_2 \preccurlyeq l$, we know that $v'_1 \approx_l v'_2$. By the hypothesis on the $g_i$, we find that $g_1(v'_1) \approx_l g_2(v'_2)$, implying $m'''_1 \approx_l m'''_2$ and $v''_1@l''_1 \approxeq_l v''_2@l''_2$ and concluding this case. Otherwise, $l'_i \not\preccurlyeq l$ for all $i$, and we conclude with Lemma A.2. □

We now show that the primitives used to define the semantics also preserve indistinguishability.

**Lemma A.7** (Label cast noninterference)**.** *If* $\bar{l}_1 \preccurlyeq \bar{l}_2$, *we have* $[\![\bar{l}_1 \preccurlyeq \bar{l}_2]\!] \approx_l [\![\bar{l}_1 \preccurlyeq \bar{l}_2]\!] : \mathsf{LIO}_{\bar{l}_1,\bar{l}_2}(1)$.

*Proof.* Suppose that we have $m_1 \approx_l m_2$, $l'$, $m'_1$, $m'_2$, $l_1$ and $l_2$ such that

$$\begin{aligned} [\![\bar{l}_1 \preccurlyeq \bar{l}_2]\!](m_1, l') &= (m'_1, 1, l_1)\\ [\![\bar{l}_1 \preccurlyeq \bar{l}_2]\!](m_2, l') &= (m'_2, 1, l_2). \end{aligned}$$

We need to show that $m_1 \vec{\uplus} m'_1 \approx_l m_2 \vec{\uplus} m'_2$ and $1@l_1 \approxeq_l 1@l_2$. The last point follows by Lemma A.4, since $1 \approx_l 1 : 1$ by definition. As for the first point, the definition of $[\![\bar{l}_1 \preccurlyeq \bar{l}_2]\!]$ implies $m'_1 = m'_2 = \emptyset$, $l_1 = l_2 = l'$ and $l' \in \downarrow\bar{l}_2$, and we conclude with the assumption $m_1 \approx_l m_2$. □

**Lemma A.8** (Cast noninterference)**.** *If* $T \preccurlyeq S$, *then* $[\![T \preccurlyeq S]\!] \approx_l [\![T \preccurlyeq S]\!]$ *for all* $l$.

*Proof.* We must show that applying a cast to indistinguishable values yields indistinguishable results. We proceed by induction on the derivation of $T \preccurlyeq S$. The cases of Unit, Bool, Label and function types follow by composition, Lemma A.7, and the induction hypotheses.

For $f = [\![\mathsf{Lab}_{\bar{l}_1}(T_1) \preccurlyeq \mathsf{Lab}_{\bar{l}_2}(T_2)]\!]$, let $v_1@l_1 \approx_l v_2@l_1$ be indistinguishable values. Assume that $f(v_1@l_1)$ and $f(v_2@l_1)$ succeed, otherwise the result is trivial. It must be the case that there are values $v'_1 = [\![T_1 \preccurlyeq T_2]\!](v_1)$ and $v'_2 = [\![T_1 \preccurlyeq T_2]\!](v_2)$, so that $f(v_i@l_1) = \mathsf{return}(v'_i@l_1)$ for all $i \in \{1, 2\}$. We conclude by combining Lemma A.6 with the induction hypothesis on $T_1$ and $T_2$. The case of reference types is similar, but without the need to analyze the inner cast. □

**Lemma A.9.** *Each primitive* $f : X$ *in Figure 14 satisfies* $f \approx_l f : X$.

*Proof. Case* unlabel. Suppose we are given labeled values $v_1@l' \approx_l v_2@l'$, memories $m_1 \approx_l m_2$ and a label $l_1$. Unfolding definitions, we have to show that $m_1 \approx_l m_2$ (obvious) and $v_1@(l' \vee l_1) \approxeq_l v_2@(l' \vee l_1)$ (which follows from Lemma A.4).

*Case* $\mathsf{get}_{\bar{l}_1,\bar{l}_2,T}$. Suppose we are given a reference $r$, memories $m_1 \approx_l m_2$, and a PC label $l_1$. We can assume that

$m_i(T^\circ, r) = v_i$ for some $v_i$, otherwise get returns an error and the result is trivial. We have to show that $v_1 @ (l_1 \vee \eta_{\mathsf{label}}) \approx_l v_2 @ (l_1 \vee \eta_{\mathsf{label}})$. By Lemma A.4, it suffices to show that $v_1 @ \eta_{\mathsf{label}} \approx_l v_2 @ \eta_{\mathsf{label}}$, which follows from the hypothesis on the memories.

*Case* $\mathsf{set}_{\tilde{l}_1, \tilde{l}_2, T}$. Suppose we are given a reference $r$, values $v_1 \approx_l v_2$, memories $m_1 \approx_l m_2$ and a PC label $l_1$. We can assume that $l_1 \preccurlyeq \eta_{\mathsf{label}}$ and $(T^\circ, r) \in \mathsf{dom}(m_i)$ for all $i$, otherwise set returns an error and the result is trivial. We have to show $1 @ l_1 \approx_l 1 @ l_1$, which is trivial, and

$$m_1 \vec{\uplus} [T^\circ, r \mapsto v_1] \approx_l m_2 \vec{\uplus} [T^\circ, r \mapsto v_2],$$

which follows from Lemma A.5.

Note that the side conditions $l_1 \preccurlyeq \eta_{\mathsf{label}}$ and $(T^\circ, r) \in \mathsf{dom}(m_i)$ are not invoked to show noninterference for this case. Instead, they are needed to ensure that set returns an element of LIO, whose properties guarantee that bind respects indistinguishability.

*Case* $\mathsf{new}_{\tilde{l}_1, \tilde{l}_2, T}$. Let $v_1 \approx_l v_2$ be values, $m_1 \approx_l m_2$ be memories, $l_1$ be a PC label, and $l_2$ be a label for the new reference. Assume that $l_1 \preccurlyeq l_2$, otherwise the two sides raise an error and the result is trivial. The hypotheses on $m_1$ and $m_2$ imply that $\min\{n \mid (T^\circ, (n, l_1, l_2)) \notin \mathsf{dom}(m_i)\}$ is the same for $i = 1$ and $i = 2$. Call this number $n$, and set $r = (n, l_1, l_2)$. The results of each execution are of the form $([T^\circ, r \mapsto v_i], r, l_1)$, and we have to show that $m_1 \vec{\uplus} [T^\circ, r \mapsto v_1] \approx_l m_2 \vec{\uplus} [T^\circ, r \mapsto v_2]$ and $r @ l_1 \approx_l r @ l_1$. The first point follows by Lemma A.5. As for the second point, it holds by Lemma A.4, since $r \approx_l r$ holds trivially.

*Case* $\mathsf{toLab}_{\tilde{l}_1, \tilde{l}_2, \tilde{l}_3}$. Suppose that we are given $l_1 \in \gamma(\tilde{l}_1)$, $f_1 \approx_l f_2 : \mathsf{LIO}_{\tilde{l}_2, \tilde{l}_3}(X)$, $m_1 \approx_l m_2$. By the definition of toLab, we can suppose that $f_i(m_i, l_2) = (m_i', v_i, l_i')$ and $l_i' \preccurlyeq l_1 \vee l_2$ for some $m_i'$, $v_i$ and $l_i'$, with $i \in \{1, 2\}$. We have to show that $m_1 \vec{\uplus} m_1' \approx_l m_2 \vec{\uplus} m_2'$ and $v_1 @ l_1 @ l_2 \approx_l v_2 @ l_1 @ l_2$. By the hypothesis on the $f_i$, we know that the first condition holds, and also that $v_1 @ l_1' \approx_l v_2 @ l_2'$. If $l_2 \preccurlyeq l$ and $l_1 \preccurlyeq l$, we know that $l_1'$ and $l_2'$ are below $l$ as well. Hence, $v_1 \approx_l v_2$, and we conclude by applying Lemma A.4. $\square$

Taken together, these lemmas lead to our main result.

**Theorem 5.1** (Noninterference). *If* $\Gamma \vdash_{\tilde{l}_1, \tilde{l}_2} e : T$, *we have*

$$[\![e]\!] \approx_l [\![e]\!] : [\![\Gamma]\!] \xrightarrow{\text{cont}} \mathsf{LIO}_{\tilde{l}_1, \tilde{l}_2}([\![T]\!]).$$

*Proof.* By induction on the typing derivation of $e$, combining the previous preservation results. We detail some cases here.

*Case* pcLabel. Suppose we are given memories $m_1 \approx_l m_2$ and a PC label $l_1$. By Lemmas A.4 and A.5, we know $\emptyset \approx_l \emptyset$ and $l_1 @ l_1 \approx_l l_1 @ l_1$, which concludes this case.

*Case* fun. It suffices to show that $\lambda v. [\![e]\!](s_1[x \mapsto v]) \approx_l \lambda v. [\![e]\!](s_2[x \mapsto v])$ assuming that $[\![e]\!] \approx_l [\![e]\!]$ and $s_1 \approx_l s_2$. Thus, we have to show $[\![e]\!](s_1[x \mapsto v_1]) \approx_l [\![e]\!](s_2[x \mapsto v_2])$ for all arguments $v_1 \approx_l v_2$. This follows because $s_1[x \mapsto v_1] \approx_l s_2[x \mapsto v_2]$, and by the definition of indistinguishability for functions.

*Case* app. By composition, it suffices to show that the last clause of the semantics of app respects indistinguishability. Unfolding what this means, we have to show that $s_1(f)(v_1) \approx_l s_2(f)(v_2)$ whenever $s_1 \approx_l s_2 : [\![\Gamma]\!]$, $\Gamma(f) = T \xrightarrow{\tilde{l}_1, \tilde{l}_2} S$ and $v_1 \approx_l v_2 : [\![T]\!]$. This follows by the definition of indistinguishability for $[\![T \xrightarrow{\tilde{l}_1, \tilde{l}_2} S]\!]$. $\square$

## A.4 Proving the gradual guarantees

The construction of the error approximation relations is similar to indistinguishability, but takes place in the admissible $\mathsf{CLat}_\wedge$-fibration $r : \mathsf{Appr} \to \mathsf{CPO}_\perp^{\mathsf{Type}}$ defined as follows. An object of Appr is a pair $(X, R)$ where $X \in \mathsf{CPO}_\perp^{\mathsf{Type}}$ and $(R(T \lhd S) \subseteq X(T) \times X(S))_{T \lhd S}$ is a family of chain-complete relations. A morphism of type $(X, R) \to (Y, U)$ is a family of partial functions $(f_T : X(T) \to Y(T)_\perp)_{T \in \mathsf{Type}}$ such that, for every $(x_T, x_S) \in R(T \lhd S)$, either $f_T(x_T) = f_S(x_S) = \perp$ or $f_T(x_T) = y_Y \in Y(T)$, $f_S(x_S) = y_S \in Y(S)$ and $(y_T, y_S) \in U(T \lhd S)$. The $r$ functor simply projects the $X$ component and acts as the identity on morphisms.

We lift $F$ to a functor $\hat{F}^\lhd$ on Appr:

$$\begin{tikzcd}
\mathsf{Appr}^{op} \times \mathsf{Appr} \arrow[r, "\hat{F}^\lhd"] \arrow[d, "q^{op} \times q"] & \mathsf{Appr} \arrow[d, "q"] \\
\left(\mathsf{CPO}_\perp^{\mathsf{Type}}\right)^{op} \times \mathsf{CPO}_\perp^{\mathsf{Type}} \arrow[r, "F"] & \mathsf{CPO}_\perp^{\mathsf{Type}}
\end{tikzcd}$$

whose definition is given in Figure 19, following similar conventions as in Appendix A.3. This allows us to construct $(R_D^\lhd(T \lhd S) \subseteq D(T) \times D(S))$ such that

$$\begin{aligned} &(\mathsf{fold}_T(x), \mathsf{fold}_S(y)) \in R_D^\lhd(T \lhd S) \\ \iff &(x, y) \in F_R^\lhd(R_D^\lhd, R_D^\lhd)(T \lhd S) \end{aligned}$$

which we take as the definition of the error approximation relations of Figure 19.

Having defined error approximation, we are ready to prove the gradual guarantees. We begin with a few auxiliary results that show that loosening the labels in a program does not interfere with subtyping or joins.

**Lemma A.10** (Dynamism and subtyping). *If* $\tilde{l}_1 \lhd \tilde{l}_2$, *then* $\tilde{l}_1 \preccurlyeq \tilde{l}$ *implies* $\tilde{l}_2 \preccurlyeq \tilde{l}$ *and* $\tilde{l} \preccurlyeq \tilde{l}_1$ *implies* $\tilde{l} \preccurlyeq \tilde{l}_2$. *If* $T_1 \lhd T_2$, *then* $T_1 \preccurlyeq S$ *implies* $T_2 \preccurlyeq S$ *and* $S \preccurlyeq T_1$ *implies* $S \preccurlyeq T_2$. *In particular, since all relations are reflexive,* $\lhd$ *and* $\rhd$ *are contained in* $\preccurlyeq$.

In light of this lemma, we will write $[\![T \lhd S]\!]$ (or $[\![T \rhd S]\!]$) instead of $[\![T \preccurlyeq S]\!]$ when $T \lhd S$ (or $T \rhd S$) holds. We will adopt a similar convention for label casts.

$$
\begin{aligned}
\hat{F}^{\triangleleft}_R(R^-,R^+)(T \triangleleft T) &\triangleq \{(x,x) \in X(T) \times X(T)\} \\
&\qquad (T = \mathsf{Unit}, \mathsf{Bool}, \mathsf{Label}) \\
\hat{F}^{\triangleleft}_R(R^-,R^+)(\mathsf{Ref}_{\bar{l}}(T) \triangleleft \mathsf{Ref}_{\bar{l}'}(T')) &\triangleq \{(x,x) \mid x \in X(\mathsf{Ref}_{\bar{l}}(T))\} \\
\hat{F}^{\triangleleft}_R(R^-,R^+)(\mathsf{Lab}_{\bar{l}}(T) \triangleleft \mathsf{Lab}_{\bar{l}'}(T')) &\triangleq \{(x@l, x'@l) \mid (x,x') \in R^+(T \triangleleft T')\} \\
\hat{F}^{\triangleleft}_R(R^-,R^+)(T_1 \xrightarrow{\bar{l}_1,\bar{l}_2} T_2 \triangleleft T_1' \xrightarrow{\bar{l}_1',\bar{l}_2'} T_2') &\triangleq \{(f,f') \mid \forall (x_1,x_1') \in R^-(T_1 \triangleleft T_1'), (m_1,m_1') \in G^{\triangleleft}_{\mathsf{Mem}}(R^-), l_1. \\
&\qquad (f(x_1)(m_1,l_1) = \bot \implies f'(x_1')(m_1',l_1) = \bot) \\
&\qquad \wedge \forall x_2, m_2, l_2.\ f(x_1)(m_1,l_1) = (m_2,x_2,l_2) \implies \\
&\qquad\quad \exists x_2', m_2'.\ f'(x_1')(m_1',l_1) = (m_2',x_2',l_2) \\
&\qquad\quad \wedge (x_2,x_2') \in R^+(T_2 \triangleleft T_2') \wedge (m_2,m_2') \in G^{\triangleleft}_{\mathsf{Mem}}(R^+)\} \\
G^{\triangleleft}_{\mathsf{Mem}}(R) &\triangleq \{(m_1,m_2) \mid \mathsf{dom}(m_1) = \mathsf{dom}(m_2) \\
&\qquad \wedge \forall (T,r) \in \mathsf{dom}(m_1).(m_1(T,r), m_2(T,r)) \in R(T \triangleleft T)\}
\end{aligned}
$$

**Figure 24.** Lifting of the functor $F$ for error approximation. The clause for Ref assumes that $X(\mathsf{Ref}_{\bar{l}}(T)) \subseteq X(\mathsf{Ref}_{\bar{l}'}(T'))$ when $\bar{l} \triangleleft \bar{l}'$. Strictly speaking, this is not valid for the entire domain of definition of the functor $F$, but we can show that $F$ does preserve this property, so there is no harm in assuming it holds.

**Lemma A.11** (Dynamism, joins and meets). *Let $\oplus \in \{\vee, \wedge\}$. If $\bar{l}_1 \triangleleft \bar{l}_1'$ and $\bar{l}_2 \triangleleft \bar{l}_2'$, then $\bar{l}_1 \oplus \bar{l}_2 \triangleleft \bar{l}_1' \oplus \bar{l}_2'$. If $T \triangleleft T'$, $S \triangleleft S'$, and $T \oplus S$ is well-defined, then so is $T' \oplus S'$; moreover, $T \oplus S \triangleleft T' \oplus S'$.*

**Lemma A.12.** *If $\bar{l} \triangleleft \bar{l}'$, then $\gamma(\bar{l}) \subseteq \gamma(\bar{l}')$ and $\downarrow\bar{l} \subseteq \downarrow\bar{l}'$.*

**Theorem 6.2** (Static Gradual Guarantee). *If $\Gamma \triangleleft \Gamma'$, $\bar{l}_1 \triangleleft \bar{l}_1'$, $e \triangleleft e'$, and $\Gamma \vdash_{\bar{l}_1,\bar{l}_2} e : T$, there exist $\bar{l}_2' \triangleright \bar{l}_2$ and $T' \triangleright T$ such that $\Gamma' \vdash_{\bar{l}_1',\bar{l}_2'} e' : T'$.*

*Proof.* By induction on the typing derivation, using Lemmas A.10 to A.12. ☐

Next, we cover a few properties of the error approximation relation.

**Lemma A.13.**
1. $\emptyset \triangleleft \emptyset$
2. *If $m_1 \triangleleft m_1'$ and $m_2 \triangleleft m_2'$, then $m_1 \vec{\uplus} m_2 \triangleleft m_1' \vec{\uplus} m_2'$.*
3. *If $T \triangleleft S$ and $x \triangleleft y : T \triangleleft S$, then $[T^\circ, r \mapsto x] \triangleleft [S^\circ, r \mapsto y]$.*

**Lemma A.14.** *If $\bar{l}_i \triangleleft \bar{l}_i'$ for $i \in \{1,2,3\}$, $T \triangleleft T'$ and $S \triangleleft S'$, then*

1. $\mathsf{return}_{\bar{l}_1,[\![T]\!]} \triangleleft \mathsf{return}_{\bar{l}_1',[\![T']\!]}$
2. $\mathsf{bind}_{\bar{l}_1,\bar{l}_2,\bar{l}_3,[\![T]\!],[\![S]\!]} \triangleleft \mathsf{bind}_{\bar{l}_1',\bar{l}_2',\bar{l}_3',[\![T']\!],[\![S']\!]}$

*Proof.* For return, suppose that we have values $v \triangleleft v'$, memories $m \triangleleft m'$ and a label $l$. We must show that $\emptyset \triangleleft \emptyset$ (trivial) and $v \triangleleft v'$ (which follows from the assumption).

For bind, suppose that we have computations

$$f_1 \triangleleft f_2 : \mathsf{LIO}_{\bar{l}_1,\bar{l}_2}([\![T]\!]) \triangleleft \mathsf{LIO}_{\bar{l}_1',\bar{l}_2'}([\![T']\!]),$$

functions

$$g_1 \triangleleft g_2 : [\![T]\!] \xrightarrow{\text{cont}} \mathsf{LIO}_{\bar{l}_2,\bar{l}_3}([\![S]\!]) \triangleleft [\![T']\!] \xrightarrow{\text{cont}} \mathsf{LIO}_{\bar{l}_2',\bar{l}_3'}([\![S']\!]),$$

memories $m_1 \triangleleft m_2$, and a label $l$. We must show that

$$\mathsf{bind}(f_1,g_1)(m_1,l) \triangleleft \mathsf{bind}(f_2,g_2)(m_2,l).$$

First, suppose that $\mathsf{bind}(f_1,g_1)(m_1,l) = \bot$. There are two cases to consider. If $f_1(m_1,l) = \bot$, we have $f_2(m_2,l) = \bot$ by the assumption on the $f_i$, and thus $\mathsf{bind}(f_2,g_2)(m_2,l)$ also diverges. Otherwise, it must be the case that there exist $m_1'$, $x_1$ and $l'$ such that $f_1(m_1,l) = (m_1',x_1,l')$ and $g_1(x_1)(m_1 \vec{\uplus} m_1', l_1') = \bot$. By the hypothesis on the $f_i$, we find $m_2'$ and $x_2$ such that $f_2(m_2,l) = (m_2',x_2,l')$, $m_1' \triangleleft m_2'$ and $x_1 \triangleleft x_2'$. By Lemma A.13, we find that $m_1 \vec{\uplus} m_1' \triangleleft m_2 \vec{\uplus} m_2'$. By the hypothesis on $g_i$, we find that $g_2(x_2)(m_2 \vec{\uplus} m_2', l') = \bot$, which concludes this case.

Now, suppose that $\mathsf{bind}(f_1,g_1)(m_1,l) = (m_1''', x_1', l'')$. By the definition of bind, we find $m_1'$, $x_1$, $l'$ and $m_1''$ such that

$$f_1(m_1,l) = (m_1',x_1,l') \qquad g_1(x_1)(m_1 \vec{\uplus} m_1', l') = (m_1'', x_1', l'')$$

$$m_1''' = m_1' \vec{\uplus} m_1''.$$

Once again, the hypothesis on the $f_i$ and $g_i$ combined with Lemma A.13 allow us to find $m_2' \triangleright m_1'$, $x_2 \triangleright x_1$, $m_2'' \triangleright m_1''$ and $x_2' \triangleright x_1'$ such that

$$f_2(m_2,l) = (m_2',x_2,l') \qquad g_2(x_2)(m_2 \vec{\uplus} m_2', l') = (m_2'', x_2', l''),$$

which allows us to conclude. ☐

**Lemma A.15.** *Suppose we have gradual labels $\bar{l}_1$, $\bar{l}_1'$, $\bar{l}_2$ and $\bar{l}_2'$ such that $\bar{l}_i \triangleleft \bar{l}_i'$ for all $i \in \{1,2\}$ and $\bar{l}_1 \preccurlyeq \bar{l}_2$ (and thus $\bar{l}_1' \preccurlyeq \bar{l}_2'$ by Lemma A.10). Then $[\![\bar{l}_1 \preccurlyeq \bar{l}_2]\!] \triangleleft [\![\bar{l}_1' \preccurlyeq \bar{l}_2']\!]$.*

*Proof.* Suppose that we are given a label $l_1 \in \downarrow\bar{l}_1 \subseteq \downarrow\bar{l}_1'$ and two memories $m \triangleleft m'$. If $l_1 \notin \downarrow\bar{l}_2$, we have $[\![\bar{l}_1 \preccurlyeq \bar{l}_2]\!](m,l_1) = \mathsf{error}$, and the result is trivial. Otherwise, we find $l_1 \in \downarrow\bar{l}_2 \subseteq \downarrow\bar{l}_2'$, and we can conclude because both executions result in $(\emptyset, 1, l_1)$. ☐

**Lemma A.16.** *Suppose that we have gradual types $T_1$, $T_1'$, $T_2$ and $T_2'$ such that $T_i \lhd T_i'$ for all $i \in \{1,2\}$ and $T_1 \preccurlyeq T_2$ (and thus $T_1' \preccurlyeq T_2'$ by Lemma A.10). Then $[\![T_1 \preccurlyeq T_2]\!] \lhd [\![T_1' \preccurlyeq T_2']\!]$.*

*Proof.* By unfolding definitions, we must show that related values are mapped to related results. We proceed by induction on $T_1$ and inversion on the derivations relating the types. The cases of Unit, Bool and Label are trivial, since they reduce to return. The case of function types follows by composition, the induction hypotheses, and by applying Lemma A.15.

It remains to show the result for $T_1 = \mathsf{Ref}_{\bar{l}_1}(S_1)$ and $T_1 = \mathsf{Lab}_{\bar{l}_1}(S_1)$. We focus on the second case, since the first one is similar. The remaining types are of the form

$$T_1' = \mathsf{Lab}_{\bar{l}_1'}(S_1') \qquad T_2 = \mathsf{Lab}_{\bar{l}_2}(S_2) \qquad T_2' = \mathsf{Lab}_{\bar{l}_2'}(S_2'),$$

with $S_1 \lhd S_1'$, $S_2 \lhd S_2'$, $\bar{l}_1 \lhd \bar{l}_1'$ and $\bar{l}_2 \lhd \bar{l}_2'$. Suppose that we are given related labeled values $v_1@l_1 \lhd v_1'@l_1 : \mathsf{Lab}_{\bar{l}_1}([\![S_1]\!]) \lhd \mathsf{Lab}_{\bar{l}_1'}([\![S_1']\!])$. If $l_1 \notin \downarrow\bar{l}_2$, we have $[\![T_1 \preccurlyeq T_2]\!](v_1@l_1) = \lambda(-).\,\mathsf{error}$, and the result is trivial. Otherwise, we have $l_1 \in \downarrow\bar{l}_2 \subseteq \downarrow\bar{l}_2'$, which implies

$$[\![T_1 \preccurlyeq T_2]\!](v_1@l_1) = \mathsf{do}\begin{cases} v_2 \longleftarrow [\![S_1 \preccurlyeq S_2]\!](v_1); \\ \mathsf{return}(v_2@l_1) \end{cases}$$

$$[\![T_1' \preccurlyeq T_2']\!](v_1'@l_1) = \mathsf{do}\begin{cases} v_2' \longleftarrow [\![S_1' \preccurlyeq S_2']\!](v_1'); \\ \mathsf{return}(v_2'@l_1). \end{cases}$$

We conclude by composition, applying the induction hypothesis on $S_1$, $S_2$, $S_1'$ and $S_2'$. ☐

**Corollary A.17.** *If $T_1 \lhd T_1'$ and $T_2 \lhd T_2'$, then $[\![T_1 : T_2]\!] \lhd [\![T_1' : T_2']\!]$.*

*Proof.* If $T_1 \preccurlyeq T_2$, then $T_1' \preccurlyeq T_2'$ by Lemma A.10, and the result is equivalent to $[\![T_1 \preccurlyeq T_2]\!] \lhd [\![T_1' \preccurlyeq T_2']\!]$, which follows from Lemma A.16. Otherwise, $[\![T_1 : T_2]\!] = \lambda(-).\,\mathsf{error}$, and the result is trivial. ☐

**Lemma A.18.** *Each primitive $f$ of Figure 14 satisfies $f \lhd f$ (given suitable parameters).*

*Proof.* By "given suitable parameters," we mean that each primitive is parameterized by CPOs and labels, and we must choose the parameters correctly for each side of the above relation for it to hold. This means that the labels on the left-hand side must be less dynamic than those on the right-hand side. We provide more precise statements for each case below.

*Case* unlabel. We want to show $\mathsf{unlabel}_{\bar{l}_1,\bar{l}_2} \lhd \mathsf{unlabel}_{\bar{l}_1',\bar{l}_2'}$ when $\bar{l}_i \lhd \bar{l}_i'$ for all $i$. Suppose that we have a label $l_1 \in \downarrow\bar{l}_1 \subseteq \downarrow\bar{l}_1'$, labeled values $v@l_2 \lhd v'@l_2'$ and memories $m_1 \lhd m_1'$. By definition, we have $l_2 = l_2'$ and $v \lhd v'$, and the results of executing unlabel are $(\emptyset, v, l_1 \vee l_2)$ and $(\emptyset, v', l_1 \vee l_2)$. Thus, we have to show $\emptyset \lhd \emptyset$ (trivial) and $v \lhd v'$, which follows from the assumption.

*Case* get. Given $\bar{l}_i \lhd \bar{l}_i'$ for all $i \in \{1,2\}$ and $T \lhd T'$ we must show

$$\begin{aligned}&\mathsf{get}_{\bar{l}_1,\bar{l}_2,T} \lhd \mathsf{get}_{\bar{l}_1',\bar{l}_2',T'} \\ &: \mathsf{Ref}_{\bar{l}_2} \xrightarrow{\text{cont}} \mathsf{LIO}_{\bar{l}_1,\bar{l}_1\vee\bar{l}_2}[\![T^\circ]\!] \lhd \mathsf{Ref}_{\bar{l}_2'} \xrightarrow{\text{cont}} \mathsf{LIO}_{\bar{l}_1',\bar{l}_1'\vee\bar{l}_2'}[\![T^\circ]\!].\end{aligned}$$

(Recall that $(T')^\circ = T^\circ$; cf. Lemma A.1.)

Suppose that we are given a label $l_1 \in \downarrow\bar{l}_1 \subseteq \downarrow\bar{l}_1'$, along elements

$$r \lhd r : \mathsf{Ref}_{\bar{l}_2} \lhd \mathsf{Ref}_{\bar{l}_2'} \qquad m_1 \lhd m_1' : \mathsf{Mem} \lhd \mathsf{Mem}\,.$$

If $(T^\circ, r) \notin \mathsf{dom}(m_1) = \mathsf{dom}(m_2)$, both executions return error and the result is trivial. Otherwise, $(T^\circ, r) \in \mathsf{dom}(m_1) = \mathsf{dom}(m_2)$, and the executions return

$$(\emptyset, m_1(T^\circ, r), l_1 \vee \eta_{\mathsf{label}}) \quad \text{and} \quad (\emptyset, m_1'(T^\circ, r), l_1 \vee \eta_{\mathsf{label}}).$$

We conclude using $m_1(T^\circ, r) \lhd m_1'(T^\circ, r)$, a consequence of $m_1 \lhd m_2$.

*Case* set. Given $\bar{l}_i \lhd \bar{l}_i'$ for $i \in \{1,2\}$ and $T \lhd T'$, we have to show that

$$\begin{aligned}&\mathsf{set}_{\bar{l}_1,\bar{l}_2,T} \lhd \mathsf{set}_{\bar{l}_1',\bar{l}_2',T'} \\ &: \mathsf{Ref}_{\bar{l}_1} \times [\![T^\circ]\!] \xrightarrow{\text{cont}} \mathsf{LIO}_{\bar{l}_2,\bar{l}_2}(1) \lhd \mathsf{Ref}_{\bar{l}_1'} \times [\![T^\circ]\!] \xrightarrow{\text{cont}} \mathsf{LIO}_{\bar{l}_2',\bar{l}_2'}(1),\end{aligned}$$

(As in the get case, $T$ does not have to vary.)

Suppose we are given a label $l_1 \in \downarrow\bar{l}_1 \subseteq \downarrow\bar{l}_1'$ along elements

$$r \lhd r : \mathsf{Ref}_{\bar{l}_2} \lhd \mathsf{Ref}_{\bar{l}_2'} \qquad v \lhd v' : [\![T^\circ]\!] \lhd [\![T^\circ]\!]$$

$$m_1 \lhd m_1' : \mathsf{Mem} \lhd \mathsf{Mem}\,.$$

Suppose that $(T^\circ, r) \in \mathsf{dom}(m_1) = \mathsf{dom}(m_1')$ and $l_1 \preccurlyeq \eta_{\mathsf{label}}$, otherwise the result follows because

$$\mathsf{set}(r, v)(m_1, l_1) = \mathsf{set}(r, v')(m_1', l_1) = \mathsf{error}\,.$$

By unfolding the definition of set, we must show that $[T^\circ, r \mapsto v] \lhd [T^\circ, r \mapsto v']$ (follows by Lemma A.13) and $1 \lhd 1$ (trivial).

*Case* new. We must show that $\mathsf{new}_{\bar{l}_1,\bar{l}_2,T} \lhd \mathsf{new}_{\bar{l}_1',\bar{l}_2',T'}$ at the CPOs

$$\begin{aligned}&\gamma(\bar{l}_1) \times [\![T^\circ]\!] \xrightarrow{\text{cont}} \mathsf{LIO}_{\bar{l}_2,\bar{l}_2}(\mathsf{Ref}_{\bar{l}_1}) \\ &\lhd \gamma(\bar{l}_1') \times [\![T^\circ]\!] \xrightarrow{\text{cont}} \mathsf{LIO}_{\bar{l}_2',\bar{l}_2'}(\mathsf{Ref}_{\bar{l}_1'})\end{aligned}$$

when $\bar{l}_i \lhd \bar{l}_i'$ for every $i$ and $T \lhd T'$. (As in the get case, $T$ does not need to vary.)

Suppose we are given $v \lhd v' : [\![T^\circ]\!] \lhd [\![T^\circ]\!]$, $l_i \in \downarrow\bar{l}_i \subseteq \downarrow\bar{l}_i'$ ($i \in \{1,2\}$) and memories $m_1 \lhd m_1'$. We have to show that $\mathsf{new}(l_1, v)(m_1, l_2) \lhd \mathsf{new}(l_1, v')(m_1', l_2)$. Suppose that $l_2 \preccurlyeq l_1$, otherwise both terms are equal to error and the result is trivial. In this case, since $\mathsf{dom}(m_1) = \mathsf{dom}(m_1')$, the results are of the form $([T^\circ, r \mapsto v], r, l_2)$ and $([T^\circ, r \mapsto v'], r, l_2)$ for some reference $r$, and we can conclude.

*Case* toLab. We want to show

$$\mathsf{toLab}_{\bar{l}_1,\bar{l}_2,\bar{l}_3,[\![T]\!]} \lhd \mathsf{toLab}_{\bar{l}'_1,\bar{l}'_2,\bar{l}'_3,[\![T']\!]}$$

whenever $\bar{l}_i \lhd \bar{l}'_i$ for every $i$ and $T \lhd T'$. Suppose that we have labels $l_1 \in \gamma(\bar{l}_1) \subseteq \gamma(\bar{l}'_1)$, $l_2 \in {\downarrow}\bar{l}_2 \subseteq {\downarrow}\bar{l}'_2$ and

$$f \lhd f' : \mathsf{LIO}_{\bar{l}_2,\bar{l}_3}(X) \lhd \mathsf{LIO}_{\bar{l}'_2,\bar{l}'_3}(X')$$

$$m_1 \lhd m'_1 : \mathsf{Mem} \lhd \mathsf{Mem}\,.$$

There are the following cases to consider.

1. If $\mathsf{toLab}(l_1, f)(m_1, l_2) = \bot$, we know that $f(m_1, l_2) = \bot$. This implies $\mathsf{toLab}(l_1, f')(m'_1, l_2) = f'(m'_1, l_2) = \bot$ by the above hypothesis, and the conclusion follows.
2. If $\mathsf{toLab}(l_1, f)(m_1, l_2) = \mathsf{error}$, the relation reduces to $\mathsf{error} \lhd \mathsf{toLab}(l_1, f')(m'_1, l_2)$, which also holds trivially.
3. Otherwise, $\mathsf{toLab}(l_1, f)(m_1, l_2)$ must be of the form $(m_2, v_2@l_1, l_2)$, with $f(m_1, l_2) = (m_2, v_2, l_3)$ and $l_3 \preccurlyeq l_1 \vee l_2$. Since $f \lhd f'$, it must be the case that $f(m'_1, l_2) = (m'_2, v'_2, l_3)$ with $m_2 \lhd m'_2$ and $v_2 \lhd v'_2$. We conclude that $\mathsf{toLab}(l_1, f')(m'_1, l_2) = (m'_2, v'_2@l_1, l_2)$, which implies the final result.

□

**Lemma A.19** (Unique Typing). *If* $\Gamma \vdash_{\bar{l}_1,\bar{l}_2} e : T$ *and* $\Gamma \vdash_{\bar{l}_1,\bar{l}'_2} e : T'$, *then* $\bar{l}_2 = \bar{l}'_2$ *and* $T = T'$.

*Proof.* By induction on $e$ and inversion on the typing derivations. □

**Theorem 6.3** (Dynamic Gradual Guarantee, General). *If* $\Gamma \vdash_{\bar{l}_1,\bar{l}_2} e : T$, $\Gamma' \vdash_{\bar{l}'_1,\bar{l}'_2} e' : T'$, $\Gamma \lhd \Gamma'$, $\bar{l}_i \lhd \bar{l}'_i$ $(\forall i \in \{1, 2\})$, $e \lhd e'$ *and* $T \lhd T'$, *then* $[\![e]\!] \lhd [\![e']\!] : [\![\Gamma]\!] \xrightarrow{\text{cont}} \mathsf{LIO}_{\bar{l}_1,\bar{l}_2}([\![T]\!]) \lhd [\![\Gamma']\!] \xrightarrow{\text{cont}} \mathsf{LIO}_{\bar{l}'_1,\bar{l}'_2}([\![T']\!])$.

*Proof.* By Theorem 6.2 and Lemma A.19, the typing derivation of $e'$ is entirely determined by that of $e$, plus $\Gamma'$ and $\bar{l}'_1$. We proceed by induction on the derivation of $e$ and inversion on the derivation of $e'$ for each case. Most cases follow by composition, induction hypotheses, and the results of Lemma A.18; we detail the interesting ones here.

*Case* pcLabel. Trivial: the results depend only on the PC label, which is the same on both sides.

*Case* fun. We have $e = \mathsf{fun}(x :_{\bar{l}} S.e_1)$ and $e' = \mathsf{fun}(x :_{\bar{l}'} S'.e'_1)$ with $\bar{l} \lhd \bar{l}'$, $S \lhd S'$ and $e_1 \lhd e'_1$. By the induction hypothesis, we know that $[\![e_1]\!] \lhd [\![e'_1]\!]$, implying that $[\![e_1]\!](s[x \mapsto v]) \lhd [\![e'_1]\!](s'[x \mapsto v'])$ for all environments $s \lhd s'$ and arguments $v \lhd v'$. By definition, $\lambda v.[\![e_1]\!](s[x \mapsto v]) \lhd \lambda v.[\![e'_1]\!](s'[x \mapsto v'])$, hence $[\![e]\!](s) \lhd [\![e']\!](s')$, concluding this case.

*Case* app. We have $e = e' = \mathsf{app}(f, x)$, with

$$\Gamma(f) : S_1 \xrightarrow{\bar{l}_3,\bar{l}_2} T \qquad \Gamma(x) : S_2$$

$$\Gamma'(f) : S'_1 \xrightarrow{\bar{l}'_3,\bar{l}'_2} T' \qquad \Gamma(x) : S'_2,$$

such that $\bar{l}_3 \lhd \bar{l}'_3$, $S_1 \lhd S'_1$, $S_2 \lhd S'_2$, $S_2 \preccurlyeq S_1$ and $S'_2 \preccurlyeq S'_1$. Given environments $s \lhd s'$, after performing the appropriate checks, it suffices to show that $s(f)(v) \lhd s'(f)(v')$ for $v \lhd v'$. This follows from the definition of error approximation for environments and for functions. □